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TOPICAL REVIEW

Braided affine geometry and q-analogs of wave operators

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Abstract
The main goal of this review is to compare different approaches to constructing the geometry associated with a Hecke type braiding (in particular, with that related to the quantum group $U_q(sl(n))$). We place emphasis on the affine braided geometry related to the so-called reflection equation algebra (REA). All objects of such a type of geometry are defined in the spirit of affine algebraic geometry via polynomial relations on generators. We begin by comparing the Poisson counterparts of 'quantum varieties' and describe different approaches to their quantization. Also, we exhibit two approaches to introducing q-analogs of vector bundles and defining the Chern–Connes index for them on quantum spheres. In accordance with the Serre–Swan approach, the q-vector bundles are treated as finitely generated projective modules over the corresponding quantum algebras. Besides, we describe the basic properties of the REA used in this construction and compare different ways of defining q-analogs of partial derivatives and differentials on the REA and algebras close to them. In particular, we present a way of introducing a q-differential calculus via Koszul type complexes. The elements of the q-calculus are applied to defining q-analogs of some relativistic wave operators.

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1. Introduction

By a braided geometry we mean a sort of noncommutative geometry related to a *braiding* which is defined as follows. Let V be a finite-dimensional vector space over the ground field $\mathbb{K}$ (of complex numbers $\mathbb{C}$ or real numbers $\mathbb{R}$). An operator

$$R : V^{\otimes 2} \to V^{\otimes 2} \tag{1.1}$$

is called *a braiding* provided it satisfies the following relation on the space $V^{\otimes 3}$:

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}. \tag{1.2}$$

Here we use the standard notations $R_{12} = R \otimes \mathrm{Id}$ and $R_{23} = \mathrm{Id} \otimes R$, where Id is the identity operator on the space V. In fact, relation (1.2) has the meaning of a very special representation of the Artin braid group. Besides, this relation is equivalent to the quantum Yang–Baxter equation and often is called the Yang–Baxter equation too.

Let us give a few examples of solutions to the Yang–Baxter equation. The first example is the classical flip σ which transposes any two elements $\sigma(x \otimes y) = y \otimes x$, $x, y \in V$. The second example is related to a $\mathbb{Z}_2$-graded vector space $V = V_{\bar 0} \oplus V_{\bar 1}$ where the flip is replaced by its super-analog $\sigma(x \otimes y) = (-1)^{\overline{x}\,\overline{y}} y \otimes x$. Here x and y are homogeneous elements of V and $\overline{x}, \overline{y} \in \mathbb{Z}_2$ are their parities. Emphasize that all super-flips (the classical flips included) are involutive, i.e. $\sigma^2 = \mathrm{Id}$.

Other known examples come from the quantum groups (QG) $U_q(\mathfrak{g})$ (see [CP]). Consider a finite-dimensional $U_q(\mathfrak{g})$-module V and take the image of the universal R-matrix in $\mathrm{End}(V^{\otimes 2})$. Then, the product of this image and the flip σ gives a solution to (1.2).

If $\mathfrak{g} = sl(n)$ and V is the first fundamental $U_q(sl(n))$-module ($q \in \mathbb{K}$ is a fixed non-zero number), then the corresponding braiding R satisfies the second degree equation

$$(R - q\,\mathrm{Id})(R + q^{-1}\,\mathrm{Id}) = 0. \tag{1.3}$$

In this case, the representation of the group algebra of the Artin braid group is reduced to the representation of the Hecke algebra. For this reason, a braiding R satisfying the additional condition (1.3) is called *the Hecke symmetry*.

If an algebra $\mathfrak{g}$ belongs to the series B_n, C_n or D_n, then the corresponding braiding $R : V^{\otimes 2} \to V^{\otimes 2}$, where V is also the first fundamental $\mathfrak{g}$-module, satisfies a third degree equation. We call it *a Birman–Murakami–Wenzl (or BMW) symmetry*. These BMW symmetries as well as the aforementioned Hecke symmetries are deformations of the classical flips. By using the so-called gluing procedure (see [G2, MM, GPS3]) it is possible to construct the Hecke symmetries which are deformations of the super-flips. In order to point out the braidings and symmetries which are deformations of the classical flips we call them *quasiclassical*. Note that there exists a big family of other braidings which are not deformations of either classical or super-flips.

Given a braiding R (quasiclassical or not), the following very natural question arises: which associative algebras can be connected with it? The simplest examples are q-analogs of the symmetric $\mathrm{Sym}(V)$ and skew-symmetric $\bigwedge(V)$ algebras on the space V endowed with a Hecke symmetry (1.1). They are respectively defined as follows:

$$\mathrm{Sym}_q(V) = T(V)/\langle \mathrm{Im}(q\,\mathrm{Id} - R)\rangle, \qquad \bigwedge\nolimits_q(V) = T(V)/\langle \mathrm{Im}(q^{-1}\mathrm{Id} + R)\rangle. \tag{1.4}$$

Hereafter, $T(V)$ stands for the free tensor algebra of a given space V and $\langle S\rangle$ is the two-sided ideal generated by a subset $S \subset T(V)$. As follows from the results of [G2], these algebras have *a good deformation property*. For a quasiclassical Hecke symmetry R this means that for a generic q and all positive integers k, we have

$$\dim \mathrm{Sym}_q^{(k)}(V) = \dim \mathrm{Sym}^{(k)}(V), \qquad \dim \bigwedge\nolimits_q^k(V) = \dim \bigwedge\nolimits^k(V), \tag{1.5}$$

where $\mathrm{Sym}_q^{(k)}(V)$ and $\bigwedge_q^k(V)$ are the kth-order homogeneous components of the algebras $\mathrm{Sym}_q(V)$ and $\bigwedge_q(V)$, respectively.

If R is a BMW symmetry, q-analogs of symmetric and skew-symmetric algebras of the space V can be introduced as well[3].

[3] Observe that, in general, for an arbitrary braiding $R : V^{\otimes 2} \to V^{\otimes 2}$ the 'R-analogs' of the symmetric $\mathrm{Sym}(V)$ and skew-symmetric $\bigwedge(V)$ algebras are not defined. The definition of these algebras as the quotients $T(V)/\langle \mathrm{Im}(\mathrm{Id} - R)\rangle$ and $T(V)/\langle \mathrm{Im}(\mathrm{Id} + R)\rangle$) respectively is acceptable only if 1 and -1 are the eigenvalues of R (otherwise these quotients are trivial). But, in general, even under this condition the good deformation propriety of the algebras involved (assuming R to be quasiclassical) is not ensured.

Two other well-known examples belong to the class of the so-called quantum matrix algebras. (All algebras below are assumed to be unital.) These are the RTT algebra and the reflection equation algebra (REA). The RTT algebra is an associative algebra generated by formal indeterminates T_i^j, $1 \leqslant i, j \leqslant n = \dim V$, subject to the system of relations [FRT]

$$R_{12}T_1T_2 - T_1T_2R_{12} = 0, \tag{1.6}$$

which is the compact notation for the matrix equation

$$R(T \otimes \mathrm{Id})(\mathrm{Id} \otimes T) - (T \otimes \mathrm{Id})(\mathrm{Id} \otimes T)R = 0.$$

Here $T = \|T_i^j\|$ is the $n \times n$ quantum matrix with noncommutative entries T_i^j.

The REA is another associative algebra with formal generators L_i^j, $1 \leqslant i, j \leqslant n = \dim V$ subject to the relations

$$R_{12}L_1R_{12}L_1 - L_1R_{12}L_1R_{12} = 0, \qquad L_1 = L \otimes \mathrm{Id}, \tag{1.7}$$

where $L = \|L_i^j\|$. Upon replacing L_1 in this relation by $L_2 = \mathrm{Id} \otimes L$ we get a similar algebra which will be referred to as *the REA of second type*.

Besides these two examples, there are other quantum matrix algebras associated with pairs of compatible braidings. We discuss them in section 3.

Also, note that the reflection equation (with a parameter) appeared in connection with the theory of integrable models with a boundary in [C] (see also [KS]).

If R is a quasiclassical Hecke symmetry, then the RTT algebra and the REA have the good deformation property and can be treated as two different q-analogs of the symmetric algebra $\mathrm{Sym}(\mathrm{End}(V))$, where $\mathrm{End}(V)$ stands for the vector space of endomorphisms of the space V. In this case, we denote the algebras (1.6) and (1.7) as $\mathrm{Sym}_q(\mathcal{T})$ and $\mathrm{Sym}_q(\mathcal{L})$, respectively. Here $\mathcal{T} = \mathrm{span}\big(T_i^j\big)_{i,j=1}^n$ and $\mathcal{L} = \mathrm{span}\big(L_i^j\big)_{i,j=1}^n$ are the linear spans of the corresponding generators. The q-analog $\bigwedge_q(\mathcal{T})$ (resp., $\bigwedge_q(\mathcal{L})$) of the skew-symmetric algebra $\bigwedge(\mathrm{End}(V))$ with the good deformation property can also be associated with any quasiclassical Hecke symmetry (see section 7).

Being equipped with the usual matrix coproduct, the algebras $\mathrm{Sym}_q(\mathcal{T})$ and $\mathrm{Sym}_q(\mathcal{L})$ become bialgebras (in the latter case *braided*). Besides, if R is an *even*[4] Hecke symmetry, then in each of these algebras there is a q-analog of the determinant of the matrix T or L which is the group-like element with respect to the matrix coproduct. Let $\det_q T$ (resp., $\mathrm{Det}_q L$) be this analog in the algebra $\mathrm{Sym}_q(\mathcal{T})$ (resp., $\mathrm{Sym}_q(\mathcal{L})$).

Assuming $\det_q T$ to be central ($\mathrm{Det}_q L$ is always central) we can define the quotients

$$\mathrm{Sym}_q(\mathcal{T})/\langle \det_q T - 1\rangle \qquad \text{and} \qquad \mathrm{Sym}_q(\mathcal{L})/\langle \mathrm{Det}_q L - 1\rangle.$$

These quotients are the Hopf algebras (in the latter case the *braided* Hopf algebra) and can be treated as two different deformations of the algebra $\mathbb{K}[SL(n)]$, provided R is a quasiclassical Hecke symmetry. The Hopf structure in the above quotient of RTT algebra first appeared in papers of the Leningrad mathematical school (see [FRT] and references therein) and afterwards was formalized by Drinfeld [Dr1]. The braided Hopf structure in the above quotient of the REA was discovered by Majid (see [M4] and references therein).

Note that if G is a group from the series B_n, C_n or D_n, then there exist similar deformations of the algebra $\mathbb{K}[G]$. They can also be realized as appropriate quotients of the RTT algebra and of the REA, respectively. In the following, the notation $\mathbb{K}_q[G]$ stands for the quantum deformation of the algebra $\mathbb{K}[G]$ which is the mentioned quotient of the RTT algebra.

In the present review, we review different ways of introducing quantum (braided) analogs of coordinate algebra of an affine regular variety and compare the roles of the RTT algebra and

[4] This means that it is skew invertible (see section 3) and the skew-symmetric algebra $\bigwedge_q(V)$ is finite dimensional.

the REA in braided geometry. Also, we exhibit a regular way of defining q-analogs of some relativistic wave operators (Laplace, Maxwell, Dirac ones) on a q-analog of the Minkowski space algebra.

We are mainly interested in a braided version of *affine* algebraic geometry. This means that all algebras we are dealing with are introduced by means of some polynomial relations on their generators. The coefficients of these relations analytically depend on the deformation parameter q which can be specialized. The RTT algebra, the REA, their quotients mentioned above and their skew-symmetric counterparts are examples of such algebras.

Other examples are provided by the so-called q-(quantum, braided) varieties. The corresponding algebras arise from a quantization of commutative algebras of functions on some classical varieties. A typical example of a classical variety to be quantized is an orbit $\mathcal{O} \subset \mathfrak{g}^*$ of a semisimple element $a \in \mathfrak{g}^*$ (a *semisimple orbit* for shot), where $\mathfrak{g}$ is a simple Lie algebra (our main example is $sl(n)$) corresponding to a complex connected Lie group G. Let $\mathbb{K}[\mathcal{O}]$ be the coordinate algebra of the affine variety $\mathcal{O} \subset \mathfrak{g}^*$.

Another way to realize the above coordinate algebra is based on the fact that the orbit $\mathcal{O}$ is isomorphic (as a G-set) to a coset G/G_a, where $G_a \subset G$ is the stabilizer of the element a. Via the map dual to the projection $G \to \mathcal{O}$ we can realize the space of functions on $\mathcal{O}$ as a subalgebra of $\mathbb{K}[G]$ consisting of functions such that $f(xb) = f(x)$, where $x \in G, b \in G_a$. Abusing the notations we denote this subalgebra by $\mathbb{K}[G/G_a]$.

In general, upon deforming the algebras $\mathbb{K}[\mathcal{O}]$ and $\mathbb{K}[G/G_a]$, we get non-equivalent quantum algebras. Moreover, if G/G_a is not a symmetric homogeneous space, the algebra $\mathbb{K}[G/G_a]$ has a large family of quantum deformations. Any algebra A of this family is a covariant $U_q(\mathfrak{g})$-module and its product (denoted $\circ$) is coordinated with $U_q(\mathfrak{g})$ action in the following sense:

$$X(a \circ b) = \circ(X_1(a) \otimes X_2(b)), \qquad \forall a, b \in A, \qquad X \in U_q(\mathfrak{g}) \tag{1.8}$$

where $\Delta(X) = X_1 \otimes X_2$ is the Sweedler's notation for the coproduct in the QG $U_q(\mathfrak{g})$.

In the following section, we consider semiclassical counterparts of the above quantum algebras on semisimple orbits[5]. Following [DGS] we show that on a generic semisimple orbit $\mathcal{O} \subset \mathfrak{g}^*$ there is a family of G-covariant non-equivalent Poisson structures. This family is parameterized by a variety of dimension equal to the rank of G excluding some subvarieties of smaller dimensions. However, in general these Poisson structures can be quantized only formally (i.e. the products in the corresponding quantum algebras are represented by series whose convergence is disregarded). One of such structures, the so-called reduced Sklyanin bracket, can be quantized in terms of 'quantum cosets'. More precisely, the resulting quantum algebra (denoted by $\mathbb{K}_q[G/G_a]$) is treated as a subalgebra of the algebra $\mathbb{K}_q[G]$ similarly to the classical pattern.

Nevertheless, if $G = SL(n)$, then on any semisimple G-orbit there is a Poisson pencil whose quantization can be described in the spirit of affine algebraic geometry. Moreover, it is a restriction of a Poisson pencil defined on the vector space $gl(n)^*$. The latter Poisson pencil

$$\{\,,\}_{a,b} = a\{\,,\}_{\mathrm{PL}} + b\{\,,\}_{\mathrm{REA}} \tag{1.9}$$

is generated by the linear Poisson–Lie bracket $\{\,,\}_{\mathrm{PL}}$ and the Poisson counterpart of the standard[6] REA denoted by $\{\,,\}_{\mathrm{REA}}$. It is known that the bracket $\{\,,\}_{\mathrm{PL}}$ can be restricted to any orbit in $\mathfrak{g}^*$ for any Lie algebra $\mathfrak{g}$. We call this restricted bracket the Kirillov–Kostant–Souriau (KKS) one and denote it by $\{\,,\}_{\mathrm{KKS}}$.

[5] We call the corresponding Poisson brackets G-covariant ones. We reserve the term 'a Poisson–Lie bracket' (which is often used for these brackets) for the linear Poisson bracket on the space $\mathfrak{g}^*$.

[6] In the following, we also use the term *standard* for the Hecke symmetry (and other objects) related to the QG $U_q(sl(n))$.

The fact that the bracket $\{\, ,\}_{\mathrm{REA}}$ can also be restricted to any orbit in $gl(n)^*$ was proved by Donin [D]. We would like to emphasize a great contribution to this area made by our friend Donin who passed away 4 years ago. Being an expert in deformation theory, he studied many types of Poisson brackets (those on homogeneous spaces included) and their quantizations.

Thus, the whole pencil (1.9) can be restricted to any orbit in $gl(n)^*$, and consequently, to any orbit in $sl(n)^*$. Observe that the quantum counterpart of the Poisson pencil (1.9) can be defined by polynomial relations which are a slight modification of those for the standard REA. Let us describe the quantum algebra explicitly.

Given a skew-invertible symmetry R, we call the modified REA (mREA) the algebra defined by the following system of quadratic–linear relations on its generators L_i^j:

$$R_{12}L_1R_{12}L_1 - L_1R_{12}L_1R_{12} - \hbar(R_{12}L_1 - L_1R_{12}) = 0, \tag{1.10}$$

where $L = \|L_i^j\|$. In what follows, we denote the mREA by $\mathcal{L}(R_q, \hbar)$.

Note that for the standard braiding R the algebra $\mathcal{L}(R_q, \hbar)$ transforms into the enveloping algebra $U(gl(n)_\hbar)$ as $q \to 1$. (Hereafter, $\mathfrak{g}_\hbar$ stands for the Lie algebra which differs from the Lie algebra $\mathfrak{g}$ by the factor $\hbar$ in the Lie brackets.) As a consequence, in the standard case the Poisson pencil (1.9) turns out to be the semiclassical counterpart of the two parameter algebra $\mathcal{L}(R_q, \hbar)$.

Comparing relations (1.7) and (1.10) we can also see that for the standard Hecke symmetry R the following holds $\mathrm{Sym}_q(\mathcal{L}) = \mathcal{L}(R_q, 0)$. In what follows, the algebra $\mathcal{L}(R_q, 0)$ will also be denoted by $\mathcal{L}(R_q)$. Note that the algebra $\mathcal{L}(R_q)$ is actually isomorphic to $\mathcal{L}(R_q, \hbar)$ if $q \neq \pm 1$ (see section 3). But since the isomorphism breaks at $q = \pm 1$ we prefer to distinguish these algebras and to use different notations for them.

Now, go back to the restriction of the Poisson pencil (1.9) to a given semisimple orbit $\mathcal{O}$. We want to represent the result of its quantization as a proper quotient of the two-parameter algebra $\mathcal{L}(R_q, \hbar)$. To this end, we have to find 'the coordinate ring of the corresponding quantum orbit'. This problem is rather subtle if the orbit $\mathcal{O}$ is not generic, i.e., the eigenvalues of the corresponding element $a \in \mathcal{O}$ are not pairwise distinct. As was observed in [DM], the degeneracy of the eigenvalues disappears after quantization (see also [GS2], remark 18). We do not consider such orbits here. Our main examples—the hyperboloids (spheres)—are the generic orbits and, at the same time, these orbits are symmetric due to their low dimension.

Note that any symmetric orbit $\mathcal{O}$ possesses an additional property: all G-covariant brackets on such an orbit $\mathcal{O}$ originate from the restriction of the Poisson pencil (1.9) to it. Thus, in this case any quantum coset algebra $\mathbb{K}_q[G/G_a]$ can also be realized as a quotient of the mREA. For example, the quantum sphere algebra can be introduced as a quantum coset $\mathbb{K}_q[SU(2)/H]$ and as a quotient of the mREA (see section 6). Note that the popular Podleś quantum sphere algebra [P1] is just such a quotient that is written in terms of generators different from ours and endowed with an involution (conjugation) possessing classical properties. However, contrary to the quantum hyperboloid algebra, the Podleś quantum sphere cannot be realized as a real algebra even if the parameter q is real. We discuss the problem of an appropriate definition of an involution in a quantum algebra in sections 5 and 6.

Nevertheless, even on symmetric orbits different ways of introducing quantum algebras give rise to different types of 'quantum geometry'. In section 6, we compare these approaches on an example of a quantum sphere (hyperboloid). In particular, we describe two ways of constructing quantum analogs of line bundles and computing the Chern–Connes index for them. Recall that according to the Serre–Swan approach any vector bundle over a regular affine algebraic variety or a smooth compact one can be realized as a projective module over its coordinate algebra (all projective modules are assumed to be finitely generated). The Chern–Connes index $\mathrm{Ind}(\pi, e)$ is introduced via a pairing of a representation

π of a given noncommutative algebra A and a projective A-module $M \cong eA^{\oplus n}$ (or $M \cong A^{\oplus n}e$), where $e \in \mathrm{Mat}_n(A)$ is the corresponding idempotent. This pairing is defined as follows:

$$\mathrm{Ind}(\pi, e) = \mathrm{Tr}(\pi(\mathrm{Tr}(e))). \tag{1.11}$$

In [HM] a family of idempotents over the algebra $\mathbb{K}_q[SU(2)/H]$ was constructed. Besides, one of these idempotents was paired with an infinite-dimensional representation of the quantum sphere algebra taken from [MNW]. The quantum index thus calculated equals the value of the classical index on the corresponding line bundle.

Another approach was suggested in [GLS2] (also, see [GS2]) where a family of idempotents over a q-hyperboloid algebra was constructed, the q-hyperboloid being realized as a quotient of the algebra $\mathcal{L}(R_q, \hbar)$. In contrast with [HM] these idempotents were constructed via a series of the Cayley–Hamilton identities valid for some matrices with entries from the mREA (in particular, for the matrix L entering the definition of the mREA). Such type identities were found in [GPS1, GPS2] for the quantum matrix algebras associated with a large class of general type Hecke symmetries. However, only for the REA (modified or not) the matrix powers in the Cayley–Hamilton identities have the usual sense (see section 3).

The second ingredient coming in the index formula is a representation. The problem of constructing representations of the REA (mainly, in the standard case) was considered in a series of papers (see, for example [K, Mu1, KSS, DKM, DS, S]). Note that in the standard case, the REA has some specific properties simplifying the problem (see section 5). A general approach to constructing the finite-dimensional, R-invariant[7] (or equivariant) representations was suggested in [GPS3]. In that paper, a quasitensor category (called Schur–Weyl) of the REA representations was defined for the REA associated with a skew-invertible general type Hecke symmetry R.

An important peculiarity of this category is a modification of the notion of the trace. For the quantum matrices the usual trace must be replaced by the categorical (or quantum) trace Tr_R (see sections 3 and 4 for detail). In particular, this new trace enables us to define the procedure of *sl-reduction* for the REA and its representations. In a sense, this procedure is analogous to the classical passage from $U(gl(n))$ to $U(sl(n))$ which is a motivation for the term 'sl-reduction'.

Given a representation of the algebra $\mathcal{L}(R_q, \hbar)$ belonging to the Schur–Weyl category and a projective module from the aforementioned family, the q-index is defined in [GLS2] by a formula analogous to (1.11) but both usual traces coming in it are replaced by the quantum trace Tr_R. As a result, the q-index equals a q-integer. This is an intrinsic property of the braided affine geometry: all the numerical characteristics (dimensions, indexes, etc) of its objects (algebras, modules, etc) become q-numbers.

Another basic feature of the braided affine geometry is a modification of the notion of a Lie algebra and a vector field. In the $U_q(\mathfrak{g})$ case the problem of defining a quantum (braided) Lie algebra can be formulated as follows. We look for a deformation of the Lie bracket $[\,,\,] : \mathfrak{g}^{\otimes 2} \to \mathfrak{g}$ such that the deformed bracket $[\,,\,]_q$ is a $U_q(\mathfrak{g})$-covariant map (we assume the space $\mathfrak{g}$ to be endowed with a $U_q(\mathfrak{g})$ action which is a deformation of the usual adjoint one) and the corresponding *enveloping algebra* is a quadratic–linear one and it possesses a *good deformation property*. This means that it is canonically isomorphic to its associated graded algebra[8] and the latter one has a good deformation property in the sense of definition based

[7] In the $U_q(sl(n))$ case, this means that the maps $\mathcal{L}(R_q, \hbar) \to \mathrm{End}(V)$ are $U_q(sl(n))$-morphisms.

[8] We refer the reader to [PP] where a general form of the Jacobi identity and that of the PBW theorem ensuring such an isomorphism are presented. Note that this Jacobi identity has nothing in common with that from section 5 which enables us to define the adjoint representation.

on formula (1.5). Also, we are interested in finding q-analogs of axioms of usual (or super-) Lie algebras.

There are known numerous attempts [W, DGHZ, DGG, LS, M2, GM] to define the quantum (braided) analogs of a Lie algebra (often without requiring the good deformation property for the 'enveloping algebra'). It turned out [GPS3] that for classical simple Lie algebras the braided deformations with the desired properties exist only for the Lie algebra $\mathfrak{g} = sl(n)$. Also, such a deformation can be defined for the general linear algebra $\mathfrak{g} = gl(n)$. Moreover, a braided analog of the Lie algebra $gl(n)$ can be associated with any skew-invertible Hecke symmetry R and the role of the corresponding enveloping algebra is played by the mREA related to the symmetry R. Note that although the mREA had been known for a long time, a 'braided Lie algebra' was extracted from this algebra only in [GPS3].

By using this braided Lie algebra we can define a q-analog of the adjoint action on the space $\mathcal{L}$ in the natural way: $\mathrm{ad}_q x(y) = [x, y]_q, \forall x, y \in \mathcal{L}$. In the $U_q(sl(n))$ case, it is a deformation of the usual adjoint action. In order to define braided analogs of vector fields arising from the usual adjoint action we should extend this q-adjoint action to the higher components of the algebra $\mathcal{L}(R_q)$ (or of $\mathcal{SL}(R_q)$ if we deal with an sl-reduced algebra). For usual or super-algebras this can be done by means of the coproduct acting on the generators in the additive way: $\Delta(X) = X \otimes 1 + 1 \otimes X$. This leads to the classical (or super-) Leibnitz rule for the vector fields. The above coproduct (properly extended to higher degree elements) is still valid for the mREA related to an involutive symmetry $R : R^2 = \mathrm{Id}$.

However, if a symmetry R coming in the definition of the REA is not involutive, this method of constructing the 'braided vector fields' fails. Fortunately, in the algebra $\mathcal{L}(R_q, 1)$ (we put $\hbar = 1$ for the sake of concreteness, since algebras $\mathcal{L}(R_q, \hbar)$ are isomorphic for all $\hbar \neq 0$) there exists a coproduct which endows it with a *braided bialgebra structure*. Note that in the $U_q(sl(n))$ case it is also a deformation of the classical coproduct. Thus, using this coproduct we can define the braided vector fields on the algebra $\mathcal{L}(R_q)$ or its sl-reduced counterpart $\mathcal{SL}(R_q)$ which are analogs of the vector fields arising from the usual adjoint action.

We call these braided vector fields *tangent* since their classical counterparts are tangent to orbits in $sl(n)^*$. In the case $n = 2$ the braided tangent vector fields are subject to the relation analogous to $xX + yY + zZ = 0$ which is valid for the infinitesimal rotations X, Y, Z in the space $\mathbb{R}^3$, x, y, z being $\mathbb{R}^3$ coordinates. Note that the vector fields X, Y, Z have the meaning of the angular momentum components.

In the present review, we use a different method of defining the tangent braided vector fields. This method is based on a conjecture that any element $f \in \mathcal{L}(R_q)$ or $f \in \mathcal{SL}(R_q)$ has a canonical (i.e. completely q-symmetrized) form. If it is so, the action of a braided vector field on the element f can be defined without using any form of the Leibnitz rule. Namely, it suffices to apply the braided vector field to the first factors of the summands constituting the canonical form of f (with a subsequent renormalization). This method is close to the technique used in [G3] for constructing Koszul type complexes (see section 7). Note that these complexes are usually employed in the theory of quadratic algebras [Ma, PP]. In a similar manner, using the aforementioned conjecture we introduce braided analogs of partial derivatives.

In the last section, we use the braided analogs of the partial derivatives and the tangent vector fields in order to define q-analogs of basic wave operators on the q-Minkowski space and q-hyperboloid algebras. A q-analog of the Minkowski space algebra was introduced in the early 1990s in [CSSW1, CSSW2, SWZ, OSWZ]. Initially, this algebra was defined via q-analogs of spinors. Lately, it was treated as a particular case of the REA [M3, Me1, MMe, AKR].

In section 8, we analyze other possible candidates for the role of the q-Minkowski space algebra assuming them to be quadratic and $U_q(sl(2))$-covariant[9]. We introduce a *truncated* REA $\mathbb{K}_q[\mathbb{R}^4]$ which differs slightly from (1.7) and in the $U_q(sl(2))$ case we treat it as the q-Minkowski space algebra.

Also, q-analogs of the Laplace[10] and Dirac operators on the q-Minkowski space and quantum sphere algebras were considered in a number of papers [Me2, BK, P4, PS] and others. In fact, both operators arise from the quadratic Casimir element $\mathrm{Cas} = \mathrm{Tr}_q L^2$ which is central in the algebra $\mathbb{K}_q[\mathbb{R}^4]$. Passing to the sl-reduced algebra $\mathbb{K}_q[\mathbb{R}^3]$ we get the reduced Casimir element Cas_{sl}. Then we obtain the q-Laplace operator on the algebra $\mathbb{K}_q[\mathbb{R}^3]$ (resp., $\mathbb{K}_q[\mathbb{R}^4]$) by replacing the algebra generators in the Casimir element Cas_{sl} (resp., Cas) by appropriate q-derivatives. In order to get the q-Laplace operator on the q-sphere (q-hyperboloid) we replace the generators of the algebra $\mathbb{K}_q[\mathbb{R}^3]$ in Cas_{sl} by braided analogs of some tangent vector fields. The element Cas_{sl} is also involved in the construction of q-Dirac operator on the algebras in question (see section 8).

Particular cases of such tangent vector fields are infinitesimal (hyperbolic) rotations whose q-analogs are often introduced via their relations with QG $U_q(sl(2))$ in the spirit of [FRT] or [LS] (we discuss such an approach in section 4). In our construction of q-derivatives or braided tangent vector fields we use a technique of Koszul type complexes which is applicable in a much more general context. An advantage of this approach becomes more evident when we pass to the quantum sphere (hyperboloid) on which the analogs of tangent and cotangent bundles are realized as one-sided projective modules. These modules play the crucial role in our defining the q-Maxwell operator on a quantum hyperboloid.

The first attempt to construct an algebra of differential forms on quantum spheres is due to Podleś [P2, P3]. However, his approach, based on the Leibnitz rule, leads to an algebra with non-classical dimensions on a generic quantum sphere. There is known another approach to constructing the differential calculus on a quantum sphere which uses q-analogs of complex coordinates z and $\bar{z}$ in the spirit of *projective* geometry (see, for instance [CHZ, SSV]).

Restricting ourselves to the *braided affine* geometry we disregard this approach as well as that based on C^*-algebras which involves a big amount of functional analysis. We refer the reader to the paper [A] which reviews some aspects of quantum geometry essentially based on the RTT algebra and using different tools of noncommutative geometry due to Connes. In contrast, our approach is completely algebraical: all 'varieties' are braided affine, all representations are finite dimensional. Mainly dealing with the REA algebra, playing the crucial role in braided affine geometry, we exhibit recent results and constructions which are covered neither by the review [M3] nor by the monograph [M4].

To complete the Introduction, we would like to mention the approach based on the so-called Heisenberg double and going back to [W]. It deals with the algebra $\mathbb{K}_q[GL(n)]\otimes\bigwedge_q(\mathcal{L})$ treated as a q-differential algebra on the group $GL(n)$. The difficulties of such a differential calculus are analyzed in numerous papers. We refer the reader to the recent paper [IP2] where an application of this approach to the so-called quantum top is exhibited.

Our paper is organized as follows. In the following section, we describe semiclassical (i.e. Poisson) counterparts of quantum algebras related to standard Hecke symmetries. In section 3, we recall a general method of introducing the quantum matrix algebras (the REA and RTT algebras included) which is based on the notion of compatible braidings. Also, there

[9] So, according to the common viewpoint the complex QG $U_q(sl(2))$ is treated to be the quantum analog of the Lorenz group. In this connection, we would like to mention an original approach of [Dob] where a q-Minkowski space is $SU_q(2,2)$-covariant.

[10] Note that we employ the term 'Laplace operator' in a large sense by admitting that the metric coming in its definition can be indefinite.

we reproduce the Cayley–Hamilton identity valid for the quantum matrix algebras related to the Hecke symmetries of general type and we discuss some special properties of the REA.

In section 4, we compare the methods of constructing the (m)REA representation theory in general and in the standard cases. Also, we exhibit the coproduct which plays the central role in constructing the aforementioned Schur–Weyl category. In section 5, we present a treatment of the mREA as an enveloping algebra of a *braided Lie algebra* and discuss the problem of defining an involution (conjugation) in this 'enveloping algebra'.

In section 6, we compare different ways of defining q-analogs of line bundles on a quantum sphere (hyperboloid) and the Chern–Connes index for them. In section 7, we introduce some elements of differential calculus on quantum algebras based on the Koszul type complexes. In section 8, we apply this technique in order to define the braided analogs of partial derivatives and other vector fields. Then we introduce the q-analogs of basic wave operators on the q-Minkowski space and q-hyperboloid space algebras.

2. Poisson counterparts of quantum varieties

In this section, we consider Poisson counterparts of quantum varieties. By a quantum variety algebra we mean a $U_q(\mathfrak{g})$-covariant algebra which is a deformation of (function algebra of) a usual homogeneous G-space. Here G is a classical matrix connected complex group (so, $\mathbb{K} = \mathbb{C}$), $\mathfrak{g}$ is its Lie algebra and $U_q(\mathfrak{g})$ is the corresponding quantum group. Besides, we assume that a triangular decomposition of the Lie algebra $\mathfrak{g}$ is fixed.

We consider a homogeneous G-space of the form G/G_Γ where G_Γ is *the Levi subgroup* corresponding to a subset Γ of the set of simple positive roots. The Lie algebra $\mathfrak{g}_\Gamma = \mathrm{Lie}(G_\Gamma)$ (called *the Levi subalgebra*) is generated by the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and by the root vectors $E_{\pm\alpha}$ corresponding to the roots $\pm\alpha$ for all $\alpha \in \Gamma$. Besides, we assume $\langle E_\alpha, E_{-\alpha}\rangle = 1$ where $\langle\, ,\, \rangle$ is the pairing defined by the Killing form.

The coset G/G_Γ is isomorphic (as a G-set) to an orbit $\mathcal{O} \subset \mathfrak{g}^*$ of a semisimple element of the space $\mathfrak{g}^*$ (*a semisimple orbit* in short). Consequently, we identify the algebra $\mathbb{K}[\mathcal{O}]$ and a subalgebra in the algebra $\mathbb{K}[G]$.

Consider an example of the above isomorphism for the matrix group $G = SL(2)$

$$L \in SL(2) \Leftrightarrow L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad - bc = 1, \qquad a, b, c, d \in \mathbb{K}.$$

In this simple case the only Levy subgroup different from $SL(2)$ itself is the Cartan subgroup H consisting of all diagonal matrices with the unit determinant. We put $\sigma(L) = (x, h, y)$ where $(x = ab, h = -ad - bc, y = -cd)$. Note that $\sigma(L) = \sigma(Lg)$ for $g \in H$. Thus, the functions x, y, z are defined on the coset $SL(2)/H$. With respect to the left $SL(2)$-action $L \to gL$, $g \in SL(2)$, the space of vectors (x, h, y) becomes a spin 1 $SL(2)$-module. We treat the vector (x, h, y) as an element of $sl(2)^* \cong sl(2)$. Since the quantity $h^2 + 4xy = (ad - bc)^2 = 1$ is stable under the left $SL(2)$-action, the image of the map σ belongs to a hyperboloid passing through the point $(0, 1, 0)$.

Now, consider the Sklyanin bracket $\{\, ,\, \}_{\mathbb{K}[G]}$ defined on the space $\mathbb{K}[G]$ as follows:

$$\{f, g\}_{\mathbb{K}[G]} = \circ\left(\rho_l^{\otimes 2}(r_-)(f \otimes g) - \rho_r^{\otimes 2}(r_-)(f \otimes g)\right), \qquad f, g \in \mathbb{K}[G], \tag{2.1}$$

where $\circ$ is the ordinary pointwise product in the algebra $\mathbb{K}[G]$, $r_- = (r_{12} - r_{21}) \in \mathfrak{g}^{\otimes 2}$ is the skew-symmetrized classical r-matrix (up to a factor 1/2), and ρ_l (resp., ρ_r) is the representation $\mathfrak{g} \to \mathrm{Vect}(G)$ of the algebra $\mathfrak{g}$ by the right-invariant (resp., left-invariant) vector fields. Namely,

$$\rho_l(X) f(a) = \partial_t f(\mathrm{e}^{tX} a)|_{t=0}, \qquad \rho_r(X) f(a) = \partial_t f(a\mathrm{e}^{-tX})|_{t=0} \quad \forall X \in \mathfrak{g}, a \in G.$$

We assume the classical r-matrix r to be chosen in such a way that

$$r_- = \sum_{\alpha\in\Omega^+}(E_\alpha \otimes E_{-\alpha} - E_{-\alpha}\otimes E_\alpha),$$

where Ω^+ is the set of positive roots and the symmetrized part $r_+ = (r_{12} + r_{21}) \in \mathfrak{g}^{\otimes 2}$ is the *split Casimir element*

$$r_+ = \sum_{\alpha\in\Omega^+}(E_\alpha \otimes E_{-\alpha} + E_{-\alpha}\otimes E_\alpha) + \cdots,$$

where we omit the terms belonging to $\mathfrak{h}^{\otimes 2}$.

It is well known that the Sklyanin bracket is a Poisson one and it is compatible with the standard matrix coproduct in the space $\mathbb{K}[G]$ in the sense of formula (2.2) where we put $M = G$.

If M is a homogeneous G-space, the space $\mathbb{K}[M]$ can be equipped with a coaction $\Delta : \mathbb{K}[M] \to \mathbb{K}[M] \otimes \mathbb{K}[G]$. We say that a Poisson bracket $\{\,,\}_{\mathbb{K}[M]}$ defined in the space $\mathbb{K}[M]$ is *G-covariant* if

$$\Delta\{f, g\}_{\mathbb{K}[M]} = \{\Delta(f), \Delta(g)\}_{\mathbb{K}[M]\otimes\mathbb{K}[G]}, \qquad \forall f, g \in \mathbb{K}[M] \tag{2.2}$$

where the bracket in the right-hand side is defined via the product of the Poisson structures.

A Poisson bracket on the space $\mathbb{K}[M]$ is called *G-invariant* if the operator $f(x) \to f(gx)$, $g \in G$ commutes with the bracket. It is easy to see that the sum of a G-invariant bracket and a G-covariant one is again a G-covariant bracket.

Remark 1. Hereafter, we use the notion of a 'bracket' in a large sense. By a bracket we mean an operator $\{\,,\} : \mathbb{K}[M]^{\otimes 2} \to \mathbb{K}[M]$ which is bilinear, skew-symmetric and satisfying the Leibnitz rule (the Jacobi identity is not required). The notion of G-covariance (invariance) can be extended to such 'brackets'.

Consider an example of G-covariant Poisson bracket on a given homogeneous G-space $\mathcal{O}$, namely, the Sklyanin bracket reduced to the space $\mathbb{K}[\mathcal{O}]$. We denote this bracket $\{\,,\}^{\mathcal{O}}$. The fact that this bracket is well defined follows from [DGS] (see also considerations below).

Write the Sklyanin bracket as a difference of two terms

$$\{\,,\}_{\mathbb{K}[G]} = \{\,,\}_{\text{left}} - \{\,,\}_{\text{right}} \tag{2.3}$$

where

$$\{f, g\}_{\text{left}} = \circ\rho_l^{\otimes 2}(r_-)(f \otimes g), \qquad \{f, g\}_{\text{right}} = \circ\rho_r^{\otimes 2}(r_-)(f \otimes g).$$

Being reduced to the space $\mathcal{O}$, the bracket $\{f, g\}_{\text{left}}$ becomes[11]

$$\{f, g\}^{\mathcal{O}}_{\text{left}} = \sum_{\alpha\in\Omega^+}(E_\alpha(f)E_{-\alpha}(g) - E_{-\alpha}(f)E_\alpha(g)),$$

where the elements $E_{\pm\alpha}$ are treated to be vector fields naturally defined on the homogeneous space G/G_Γ which is isomorphic to $\mathcal{O}$. As for the reduced bracket $\{\,,\}^{\mathcal{O}}_{\text{right}}$, it becomes G-invariant. Nevertheless, in general the above left and right brackets are not Poisson ones, since none of them satisfies the Jacobi identity.

As was shown in [GP], on a semisimple orbit $\mathcal{O} \subset \mathfrak{g}^*$ the bracket $\{\,,\}^{\mathcal{O}}_{\text{left}}$ becomes a Poisson one iff the orbit $\mathcal{O}$ is symmetric[12]. In the case $G = SL(n)$ a semisimple orbit is symmetric iff

[11] If r is an arbitrary element of $\bigwedge^2(\mathfrak{g})$ and $\mathfrak{g}$ is represented in the space of functions on a homogeneous space M by vector fields $\rho : \mathfrak{g} \to \text{Vect}(M)$, the expression $\{f, g\}_r = \circ\rho^{\otimes 2}(r)(f \otimes g)$ is often called the r-matrix bracket. Thus, the bracket $\{f, g\}^{\mathcal{O}}_{\text{left}}$ is just of this type.

[12] In [GP] the orbits possessing this property were called the R-matrix type orbits. As was shown there, if $\mathcal{O}$ is an R-matrix type orbit, then it is semisimple or nilpotent. For nilpotent orbits a necessary and sufficient condition to be of the R-matrix type was also found.

it is the orbit of a diagonal matrix with two different eigenvalues. It is not difficult to see that $\{\ ,\ \}^{\mathcal{O}}_{\text{right}}$ and $\{\ ,\ \}^{\mathcal{O}}_{\text{left}}$ become Poisson brackets simultaneously. Indeed, the Schouten bracket of $\{\ ,\ \}^{\mathcal{O}}$ contains three terms: the Schouten bracket of $\{\ ,\ \}^{\mathcal{O}}_{\text{left}}$ with itself, that of $\{\ ,\ \}^{\mathcal{O}}_{\text{right}}$ with itself and that of these two brackets. Since the latter Schouten bracket always vanishes the former Schouten brackets vanish simultaneously (see [DGS] for detail).

Moreover, if $\mathcal{O}$ is a symmetric orbit, the bracket $\{f, g\}^{\mathcal{O}}_{\text{right}}$ equals (up to a factor) the Kirillov–Kostant–Souriau (KKS) one $\{f, g\}^{\mathcal{O}}_{\text{KKS}}$ which is the restriction of the linear Poisson–Lie bracket defined on $\mathfrak{g}^*$ to the orbit $\mathcal{O}$. This follows from the fact that in this case the space $\mathbb{K}[\mathcal{O}]$ is multiplicity free and therefore any two G-invariant brackets are proportional to each other.

Finally, on a symmetric orbit $\mathcal{O}$ there exists a Poisson pencil

$$\{\ ,\ \}^{\mathcal{O}}_{a,b} = a\{\ ,\ \}^{\mathcal{O}}_{\text{KKS}} + b\{\ ,\ \}^{\mathcal{O}}_{\text{left}}.$$

In [DGK] an attempt was undertaken to quantize this Poisson pencil on the $\mathbb{CP}^n$ type orbits (i.e. those of minimal dimension) via the generalized Verma modules. Another way of quantizing such a Poisson pencil in the spirit of the affine algebraic geometry uses quotienting the REA. We exhibit this way below.

Let us go back to the example above. Consider the reduced Sklyanin bracket on the hyperboloid $\mathcal{O} = \{(x, h, y)|h^2 + 4xy = 1\}$. Since this orbit is symmetric, the bracket $\{f, g\}^{\mathcal{O}}_{\text{right}}$ equals (up to a factor) the KKS one. In order to find this factor we compute

$$\{x, y\}^{\mathcal{O}}_{\text{right}} = \{ab, -cd\}_{\text{right}} = -(a^2d^2 - b^2c^2) = -(ad - bc)(ad + bc) = h.$$

Thus, it equals the KKS bracket.

Compute now the bracket $\{f, g\}^{\mathcal{O}}_{\text{left}} = (X(f)Y(g) - Y(f)X(g))$ where X, H, Y are the standard generators of $sl(2)$ represented by the infinitesimal hyperbolic rotations:

$$X = h\partial_y - 2x\partial_h, \qquad H = 2x\partial_x - 2y\partial_y, \qquad Y = -h\partial_x + 2y\partial_h.$$

We have

$$\{h, x\}^{\mathcal{O}}_{\text{left}} = 2hx, \qquad \{h, y\}^{\mathcal{O}}_{\text{left}} = -2hy, \qquad \{x, y\}^{\mathcal{O}}_{\text{left}} = h^2.$$

Thus, the Sklyanin bracket reduced to the hyperboloid $\mathcal{O}$ is

$$\{h, x\}^{\mathcal{O}} = 2(h - 1)x, \qquad \{h, y\}^{\mathcal{O}} = -2(h - 1)y, \qquad \{x, y\}^{\mathcal{O}} = h(h - 1).$$

As for the above Poisson pencil we have

$$\{h, x\}^{\mathcal{O}}_{a,b} = a(2x) + b(2hx), \qquad \{h, y\}^{\mathcal{O}}_{a,b} = a(-2y) + b(-2hy), \qquad \{x, y\}^{\mathcal{O}}_{a,b} = ah + bh^2.$$

It is easy to see that this bracket is Poisson on the whole space $sl(2)^*$ too.

Our calculations are analogous to those from [Sh] (see the appendix) where the compact form of the group, i.e. $SU(2)$, was considered, but a concrete form of a given complex group is somewhat pointless. In [KRR], it was shown that the Sklyanin bracket reduced to a semisimple orbit $\mathcal{O}$ is compatible with the KKS one iff the orbit is symmetric. This entails that if an orbit $\mathcal{O}$ is not symmetric then the brackets $\{\ ,\ \}^{\mathcal{O}}_{\text{right}}$ and $\{\ ,\ \}^{\mathcal{O}}_{\text{KKS}}$ are not compatible (i.e. their Schouten bracket does not vanish).

Note that the authors of [KRR] also deal with compact forms of groups and corresponding orbits. However, as we said above this does not affect the result.

Nevertheless, if a given orbit $\mathcal{O} \subset \mathfrak{g}^*$ is not symmetric, the family of G-covariant Poisson brackets becomes larger. Let $\mathcal{O}$ be a generic orbit. This means that G_Γ equals the Cartan subgroup H. Let us describe the family of G-covariant Poisson structures following [DGS].

Consider an element

$$v = \sum_{\alpha \in \Omega^+} c(\alpha)(E_\alpha \otimes E_{-\alpha} - E_{-\alpha} \otimes E_\alpha),$$

where $c(\alpha)$ is a $\mathbb{K}$-valued function on the set Ω^+ such that if $\alpha+\beta\neq 0$ then

$$c(\alpha+\beta)=\frac{c(\alpha)c(\beta)+1}{c(\alpha)+c(\beta)}. \tag{2.4}$$

Here we assume that $c(\alpha)+c(\beta)\neq 0$ for any $\alpha,\beta\in\Omega^+$ such that $\alpha+\beta\neq 0$. This condition forbids some low-dimensional subvarieties. Now, we modify the Sklyanin bracket (2.3) on G by replacing the bracket $\{\,,\}_{\mathrm{right}}$ for the following one:

$$\{f,g\}_{\mathrm{right},c}=\circ\big(\rho_r^{\otimes 2}(v)(f\otimes g)\big).$$

The bracket $\{f,g\}_{\mathrm{right},c}$ can also be reduced to the homogeneous space $\mathcal{O}\cong G/H$ and the bracket

$$\{\,,\}_c^{\mathcal{O}}=\{\,,\}_{\mathrm{left}}^{\mathcal{O}}-\{\,,\}_{\mathrm{right},c}^{\mathcal{O}}$$

is a Poisson one as was shown in [DGS].

Note that if $\lambda:\Omega^+\to\mathbb{K}$ is a linear function then $c(\lambda)=\coth(\lambda(\alpha))$ is a solution to (2.4). Thus, the variety of G-covariant Poisson brackets $\{\,,\}_c^{\mathcal{O}}$ can be identified with a vector space of dimension $\dim\mathfrak{h}$ with exception of the aforementioned low-dimensional subvarieties. Observe that the reduced Sklyanin bracket corresponds to the case $c(\alpha)=1$ for all $\alpha\in\Omega^+$.

If an orbit $\mathcal{O}$ is not generic, i.e. $G_\Gamma\neq H$, then different functions λ can give rise to the same Poisson structure on $\mathcal{O}$, i.e. in this case the map $c\to\{\,,\}_c^{\mathcal{O}}$ is not injective.

Emphasize that the bracket $\{\,,\}_{\mathrm{left}}-\{\,,\}_{\mathrm{right},c}$ is not Poisson one on G (it becomes Poisson bracket only after the reduction to $\mathcal{O}$). But it can be treated in terms of the so-called classical dynamical r-matrices (see [L] and references therein). Also, note that another but equivalent approach to the classification problem of G-covariant Poisson structures on homogeneous G-spaces was developed in [Ka].

A way of formal quantization of the Poisson brackets $\{\,,\}_c^{\mathcal{O}}$ was suggested in [DGS] where as it is usual in the framework of formal quantization, the problem of convergency of series defining the deformed product is disregarded. It would be interesting to express the bracket $\{\,,\}_c^{\mathcal{O}}$ in terms of the coordinate algebra $\mathbb{K}[\mathcal{O}]$ and to quantize it explicitly by presenting the result in the spirit of affine algebraic geometry or to show that it is not possible.

Meanwhile, on the whole vector space $gl(n)^*$ there exists a $GL(n)$-covariant Poisson structure which is the semiclassical counterpart of the REA (for this reason we denote the corresponding bracket $\{\,,\}_{\mathrm{REA}}$). Moreover, as shown in [D] the bracket $\{\,,\}_{\mathrm{REA}}$ can be restricted to any orbit $\mathcal{O}\subset gl(n)^*$ giving rise to the $GL(n)$-covariant Poisson structure on the coordinate algebra $\mathbb{K}[\mathcal{O}]$.[13]

Besides, the bracket $\{\,,\}_{\mathrm{REA}}$ is compatible with the linear Poisson–Lie bracket $\{\,,\}_{\mathrm{PL}}$ also defined on $gl(n)^*$. By direct computations it is easy to see that the semiclassical counterpart of the two parameter mREA $\mathcal{L}(R_q,\hbar)$ is just the Poisson pencil (1.9) generated by these two brackets. Thus, this Poisson pencil can be quantized explicitly in the spirit of affine algebraic geometry. By treating its quantum counterpart via the algebra $\mathcal{L}(R_q,\hbar)$ we avoid the convergency problem of the deformation quantization scheme. The restrictions of the above Poisson pencil to semisimple orbits can also be quantized in a similar manner. As a result of such a quantization we get quotients of the mREA discussed below.

The explicit form of the bracket $\{\,,\}_{\mathrm{REA}}$ reads (see [Mu1, GPS3])

$$\{f,g\}_{\mathrm{REA}}=\circ r_+^{\mathrm{l,r}}(f\otimes g)-\circ r_+^{\mathrm{r,l}}(f\otimes g)-\circ r_-^{\mathrm{ad,ad}}(f\otimes g),\qquad \forall f,g\in\mathbb{K}[gl(m)^*]. \tag{2.5}$$

Here

$$r_-=\sum_{i<j}e_i^j\otimes e_j^i-e_j^i\otimes e_i^j,\qquad r_+=\sum_{i,j}e_i^j\otimes e_j^i,$$

[13] In fact, there are two structures: one of them is considered below, another one is associated with the second type REA in a similar way.

the elements e_i^j form the standard basis of the Lie algebra $gl(n)$

$$[e_i^j, e_k^r] = \delta_k^j e_i^r - \delta_i^r e_k^j,$$

and the element $r_+ \in gl(n)^{\otimes 2}$ is the $gl(n)$ *split Casimir element*. Superscripts indicate the type of the action. Namely,

$$r_-^{\mathrm{ad,ad}}(f \otimes g) = \sum_{i<j} \left(\mathrm{ad}\, e_i^j(f) \otimes \mathrm{ad}\, e_j^i(g) - \mathrm{ad}\, e_j^i(f) \otimes \mathrm{ad}\, e_i^j(g)\right),$$

$$r_+^{\mathrm{l,r}}(f \otimes g) = \sum_{i,j} \rho_l\left(e_i^j\right) f \otimes \rho_r\left(e_j^i\right) g, \qquad r_+^{\mathrm{r,l}}(f \otimes g) = \sum_{i,j} \rho_l\left(e_i^j\right) g \otimes \rho_r\left(e_j^i\right) f.$$

Here the notation $\mathrm{ad}\, e_i^j$ stands for the vector field acting on $l_k^r \in \mathbb{K}[GL(n)]$ by the rule $\mathrm{ad}\, e_i^j\left(l_k^r\right) = \delta_k^j l_i^r - \delta_i^r l_k^j$, whereas the notations ρ_l, ρ_r have the same meaning as above.

Let us list the basic properties of the bracket $\{\, ,\}_{\mathrm{REA}}$. First of all, it is $GL(n)$-covariant. More precisely, it is a linear combination of the r-matrix bracket (the third term in (2.5)) and $GL(n)$-invariant bracket (the first two terms in (2.5)). Second, it can be restricted to the subspace $sl(n)^*$ since for all $f \in \mathbb{K}[GL(n)]$ we have $\{f, \ell\}_{\mathrm{REA}} = 0$, where $\ell = \sum l_i^i$. The restricted bracket $\{\, ,\}_{sl-\mathrm{REA}}$ is $SL(n)$-covariant.

Remark 2. The first two terms in (2.5) can be interpreted in the following way [G3]. The space $sl(n)^{\otimes 2}$ considered as an $sl(n)$-module can be decomposed into a direct sum of irreducible $sl(n)$-modules. For $n > 2$ this decomposition contains two components isomorphic to the algebra $sl(n)$. One of these components belongs to the skew-symmetric part $\bigwedge^2(sl(n))$, the other one belongs to the symmetric part $\mathrm{Sym}^{(2)}(sl(n))$. So, there exists unique (up to a factor) nontrivial $sl(n)$-morphism $\sigma : \bigwedge^2(sl(n)) \to \mathrm{Sym}^{(2)}(sl(n))$ (it identifies the corresponding $sl(n)$-components and kills all the others). On extending the morphism σ up to the algebra $\mathrm{Sym}(sl(n))$ via the Leibniz rule, we get an $SL(n)$-invariant bracket. (Note that for a Lie algebra $\mathfrak{g}$ of the series B_n, C_n or D_n no similar bracket exists since the space $\mathfrak{g}^{\otimes 2}$ is multiplicity free.) At some specific value of the factor of the morphism σ, the corresponding bracket coincides with that defined by the first two terms of (2.5) restricted to $sl(n)^*$.

The bracket $\{\, ,\}_{sl-\mathrm{REA}}$ can be restricted to any orbit $\mathcal{O} \subset sl(n)^*$ (see [D]). If such an orbit is semisimple, it is closed. Let $I_{\mathcal{O}}$ be an ideal of functions vanishing on such an orbit $\mathcal{O}$. Then $\{f, g\}_{sl-\mathrm{REA}} \in I_{\mathcal{O}}$ for any $f \in \mathbb{K}[sl(n)]^* \cong \mathrm{Sym}(sl(n))$ and $g \in I_{\mathcal{O}}$. The quantization of the Poisson pencil generated by the KKS bracket on $\mathcal{O}$ and the bracket $\{\, ,\}_{sl-\mathrm{REA}}$ restricted to the orbit can be done via a proper quotienting the standard mREA $\mathcal{L}(R_q, \hbar)$ in the spirit of affine algebraic geometry. Thus, we get the braided coordinate algebra $\mathbb{K}_q[\mathcal{O}]$ which is a deformation of the classical algebra $\mathbb{K}[\mathcal{O}]$ whereas a quantization of the restricted bracket $\{\, ,\}_{sl-\mathrm{REA}}$ alone can be realized as a quotient of the algebra $\mathcal{L}(R_q)$.

If the orbit $\mathcal{O}$ is generic, the ideal $I_{\mathcal{O}}$ is generated by the elements $\mathrm{Tr}L^k - a_k$ with appropriate $a_k \in \mathbb{K}$, $2 \leqslant k \leqslant n$ (since $\mathrm{Tr}(L) = 0$). Here L is the classical analog of the matrix L coming in the definition of the algebra $\mathcal{L}(R_q)$, i.e. it satisfies the defining relation of the REA but with R replaced by the usual flip. Then we get the algebra $\mathbb{K}_q[\mathcal{O}]$ as a quotient of the algebra $\mathcal{L}(R_q, \hbar)$ or $\mathcal{L}(R_q)$ over an ideal generated by the elements $\mathrm{Tr}_R(L^k) - a_k$, $2 \leqslant k \leqslant n$, where Tr_R is quantum trace introduced below. If an orbit $\mathcal{O}$ is not generic the problem of finding the 'quantum coordinate algebra' $\mathbb{K}_q[\mathcal{O}]$ is more subtle (see [DM]).

Emphasize that for other simple Lie algebras there exist orbits on which the bracket $\{\, ,\}_c^{\mathcal{O}}$ is compatible with the KKS one for some c (such orbits were called *good* in [DGS]). However, these brackets and the corresponding pencils were quantized in [DGS] via the deformation quantization scheme only. It is not known whether it is possible to define the corresponding quantum algebra by a finite set of polynomial relations among their generators (at least for the matrix groups).

Let us give a short summary of the considerations above. If an orbit $\mathcal{O} \subset sl(n)^*$ is symmetric, the family of $SL(n)$-covariant brackets is just the Poisson pencil generated by the KKS bracket and the r-matrix one. Quantization of any Poisson bracket from this pencil can always be realized as a proper quotient of the algebra $\mathcal{L}(R_q, \hbar)$. Thus, we get a *braided affine variety*. If a semisimple orbit $\mathcal{O}$ is not symmetric, the quantization method depends on a given bracket on $\mathbb{K}[\mathcal{O}]$. However, the previous method is still valid, provided such a bracket is a linear combination of the KKS bracket and that $\{\ ,\ \}_{sl-\mathrm{REA}}$.

3. Quantum matrix algebras

The REA and the RTT algebra defined above are particular examples of the so-called *quantum matrix algebras*. First examples of these appeared in works [FM] and [H]. A general definition and extensive studying of the algebraic structure were given in [IOP].

The definition of the quantum matrix algebra is based on the pair of the compatible braidings which are defined below.

Definition 3. *Let V be a finite-dimensional vector space over the ground field $\mathbb{K}$, $\dim V = N$, and let $R, F \in \mathrm{End}(V^{\otimes 2})$ be two invertible operators. An ordered pair $\{R, F\}$ is called a pair of compatible braidings if the following conditions are satisfied:*

- *Both operators R and F obey the quantum Yang–Baxter equation (1.2).*
- *The operators R and F satisfy the compatibility conditions*

$$R_{12}F_{23}F_{12} = F_{23}F_{12}R_{23}, \qquad F_{12}F_{23}R_{12} = R_{23}F_{12}F_{23}. \tag{3.1}$$

Given a pair of compatible braidings $\{R, F\}$ we shall additionally assume the braidings to be strictly skew invertible.

Definition 4. *An operator R is said to be skew-invertible if there exists an operator $\Psi^R \in \mathrm{End}(V^{\otimes 2})$ such that*

$$\mathrm{Tr}_{(2)} R_{12}\Psi^R_{23} = P_{13} = \mathrm{Tr}_{(2)} \Psi^R_{12} R_{23}, \tag{3.2}$$

where the subscript in the notation of the trace shows the index of the space V, where the trace is applied (the enumeration of the factors in the tensor product is taken as follows: $V^{\otimes k} = V_1 \otimes V_2 \otimes \cdots \otimes V_k$). The skew-invertible operator R is strictly skew invertible if the operator

$$C = \mathrm{Tr}_{(2)} \Psi^R_{12} \tag{3.3}$$

is invertible.

Let us note that for a strictly skew-invertible braiding R the operator

$$B = \mathrm{Tr}_{(1)} \Psi^R_{12} \tag{3.4}$$

is also invertible [O].

As a direct consequence of definitions of B and C we find

$$\mathrm{Tr}_{(1)} B_1 R_{12} = \mathrm{Id} = \mathrm{Tr}_{(2)} C_2 R_{12}. \tag{3.5}$$

The operators B and C play the crucial role for the structure of the quantum matrix algebras and, in particular, for the representation theory of REA.

With the matrix C one defines the R-*trace* operation $\mathrm{Tr}_R : \mathrm{Mat}_N(W) \to W$

$$\mathrm{Tr}_R(X) = \sum_{i,j=1}^{N} C_i^j X_j^i, \qquad X \in \mathrm{Mat}_N(W), \tag{3.6}$$

where W is any linear space.

Remark 5. Note that if we work with the second form of REA, the matrix C in the above definition of the R-trace should be changed for the matrix B.

Now we are ready to give a general definition of a quantum matrix algebra.

Definition 6 [IOP]. *Given a compatible pair $\{R, F\}$ of strictly skew-invertible braidings, the quantum matrix algebra (QMA) $\mathcal{M}(R, F)$ is defined to be a unital associative algebra generated by N^2 components of the matrix $M = \|M_i^j\|_{1\leqslant i,j\leqslant N}$ subject to the relations*

$$R_{12}M_{\bar{1}}M_{\bar{2}} = M_{\bar{1}}M_{\bar{2}}R_{12}, \tag{3.7}$$

where we use the following notation:

$$M_{\bar{1}} = M_1 = M \otimes \mathrm{Id}, \qquad M_{\bar{2}} = F_{12}M_{\bar{1}}F_{12}^{-1} \tag{3.8}$$

for the copies of the matrix M.

The defining relations (3.7) then imply the same type of relations for consecutive pairs of the copies of M (see [IOP])

$$R_{k,k+1}M_{\bar{k}}M_{\overline{k+1}} = M_{\bar{k}}M_{\overline{k+1}}R_{k,k+1}, \tag{3.9}$$

where $M_{\overline{k+1}} = F_{k,k+1}M_{\bar{k}}F_{k,k+1}^{-1}$.

Remark 7. As can be easily seen, the pairs $\{R, P\}$ (P being a permutation matrix) and $\{R, R\}$ are compatible pairs of braidings in the sense of definition 3. The first of them defines the RTT algebra while the second one gives rise to the REA. Besides these evident examples, there exist large multiparametric families of compatible braidings leading to other quantum matrix algebras.

From now on, we shall be interested in a subfamily of QMAs characterized by the property that in the pair $\{R, F\}$ the first component R is a skew-invertible Hecke symmetry. Recall that it means that skew-invertible matrix R additionally obeys the quadratic Hecke condition

$$(R - q\,\mathrm{Id})(R + q^{-1}\mathrm{Id}) = 0, \qquad q \in \mathbb{C}\backslash 0, \tag{3.10}$$

where the numerical parameter q is either equal to 1 or is not a root of unity of any order: $q^k \neq 1, \forall k \in \mathbb{Z}_+$.

With any Hecke symmetry R one can associate an ordered pair of integers $(m|n)$ called the *bi-rank*[14] of the symmetry R. A Hecke symmetry with the bi-rank $(m|n)$ can be considered as a generalization of the super-flip on the vector super-space $V_{(m|n)}$. Therefore, we call this Hecke symmetry (and the corresponding QMA) $GL(m|n)$ type braiding. Note, that any skew-invertible Hecke symmetry of the bi-rank $(m|n)$ is automatically strictly skew-invertible since it can be proved (see [GPS3], corollary 8) that

$$B \cdot C = q^{2(n-m)}\mathrm{Id}. \tag{3.11}$$

[14] This notion was defined in [GPS3] as follows. The Poincaré–Hilbert series of the algebra $\bigwedge_q(V)$ where V is a vector space endowed with a Hecke symmetry $R : V^{\otimes 2} \to V^{\otimes 2}$, is always a rational function. Let us assume it to be uncancelable. Then m (resp., n) is the degree of its numerator (resp., denominator).

A Hecke symmetry R realizes local (or R-matrix) representations of the A_{n-1} series Hecke algebras $\mathcal{H}_n(q)$. The detailed treatment of the structure of the Hecke algebras and their R-matrix representations can be found in the review [OP1] and references therein.

Describe now the main properties of a general $GL(m|n)$ type QMA. First of all, we can extract the commutative *characteristic subalgebra* Char(M) which is formed by the linear span of the QMA elements of the following form:

$$x(h_k) = \mathrm{Tr}_{R(1,\ldots,k)}(M_{\bar{1}}\cdots M_{\bar{k}}\rho_R(h_k)) \qquad k = 1, 2, \ldots, \tag{3.12}$$

where h_k runs over all elements of the Hecke algebra $\mathcal{H}_k(q)$ and $\rho_R : \mathcal{H}_k(q) \to \mathrm{End}(V^{\otimes k})$ is the R-matrix representation of the Hecke algebra associated with a given Hecke symmetry R. The symbol $\mathrm{Tr}_{R(1,\ldots,k)}$ means that we apply the R-trace over the spaces from the first to the kth ones.

Among elements of the characteristic subalgebra we distinguish the set of 'power sums' $p_k(M)$ and Schur symmetric functions $s_\lambda(M)$ defined respectively by the relations

$$p_0(M) := \mathrm{Tr}_R(\mathrm{Id}), \qquad p_k(M) = \mathrm{Tr}_{R(1\cdots k)}(M_{\bar{1}}\cdots M_{\bar{k}}R_{k-1}\cdots R_1), \tag{3.13}$$

where $R_i = R_{i,i+1}$ and

$$s_0(M) = 1, \qquad s_\lambda(M) = \mathrm{Tr}_{R(1\cdots k)}\big(M_{\bar{1}}\cdots M_{\bar{k}}\rho_R\big(E_\alpha^\lambda\big)\big). \tag{3.14}$$

Here $\lambda \vdash k$ is an arbitrary partition of the given integer $k \in \mathbb{Z}_+$ and E_α^λ stands for a primitive idempotent of the Hecke algebra $\mathcal{H}_k(q)$ parameterized by the Young tableau $(\lambda;\alpha)$ corresponding to the partition λ. The right-hand side of (3.14) does not actually depend on the index α of the primitive idempotent. It can be proved that the elements $p_k(M)$ generate the characteristic subalgebra [IOP] and the elements $s_\lambda(M)$ form its linear basis [GPS1].

Besides, we define *the kth powers* $M^{\bar{k}}$ of the matrix M by the following rule:

$$\begin{aligned} &M^{\bar{0}} = \mathrm{Id}, \qquad M^{\bar{1}} = M_1, \\ &M^{\bar{k}} = \mathrm{Tr}_{R(2\cdots k)}(M_{\bar{1}}\cdots M_{\bar{k}}R_{k-1}\cdots R_1), \qquad k = 2, 3, \ldots. \end{aligned} \tag{3.15}$$

Comparing definitions (3.13) and (3.15) we find the relation $p_k(M) = \mathrm{Tr}_R(M^{\bar{k}})$ which is similar to the classical one. But in general, the usual matrix product of quantum matrices does not possess the classical property, that is

$$M^{\bar{k}} \cdot M^{\bar{p}} \neq M^{\overline{k+p}}.$$

Nevertheless, it is possible to define an associative multiplication $\star$ of quantum matrices which satisfies the following remarkable properties (see [OP2]):

$$M^{\overline{k+1}} = M \star M^{\bar{k}} = M^{\bar{k}} \star M, \qquad (\mathrm{Id}\, x(M)) \star M^{\bar{k}} = M^{\bar{k}} \star (x(M)\,\mathrm{Id}),$$

where $x(M)$ is an arbitrary element of the characteristic subalgebra.

Definition (3.15) is justified by the fundamental property of any $GL(m|n)$ type QMA. Namely, in each such algebra there exists a polynomial identity in powers $M^{\bar{k}}$ of order $m+n$ generalizing the classical Cayley–Hamilton identity valid for the numerical matrices. The coefficients of the 'quantum Cayley–Hamilton polynomial' are some linear combinations of the Schur functions $s_\lambda(M)$. Moreover, the quantum Cayley–Hamilton polynomial can be presented in a factorized form as the $\star$-product of two polynomials of the orders m and n. This fact enables us to introduce the notion of the 'spectrum' of the quantum (super) matrix and distinguish the 'odd' and 'even' eigenvalues. The detailed treatment of these questions with complete proofs is given in [GPS1, GPS2].

Now we give some explicit formulae for the case of $\mathcal{M}(R,R)GL(m|n)$ type QMA (hereafter we restrict ourselves to this case). As we already mentioned above, this is nothing but the REA. In this case, the complicated $\star$-product of quantum matrices reduces to the usual matrix product and the characteristic subalgebra Char(L) is *central* in the REA, that is

$$L^{\bar{k}} \equiv L^k = L \cdot L^{k-1} = L^{k-1} \cdot L, \qquad x(L)L = Lx(L), \qquad \forall x(L) \in \mathrm{Char}(L).$$

On introducing a special notation $[m|n]_r^k$ for the partition $((n+1)^k, n^{m-k}, r) \vdash (mn+k+r)$ we can write the Cayley–Hamilton identity as follows [GPS1]:

$$\sum_{i=0}^{m+n} L^{m+n-i} \sum_{k=\max\{0,i-n\}}^{\min\{i,m\}} (-1)^k q^{2k-i} s_{[m|n]_{i-k}^k}(L) \equiv 0.$$

This identity can be presented in a remarkable factorized form. Multiplying the above formula by $s_{[m|n]}(L)$, we can rewrite the result as follows [GPS2]:

$$\left(\sum_{k=0}^{m} (-q)^k L^{m-k} s_{[m|n]_0^k}(L)\right) \cdot \left(\sum_{r=0}^{n} q^{-r} L^{n-r} s_{[m|n]_r^0}(L)\right) \equiv 0.$$

Hereafter, the notation $[m|n] = [m|n]_0^0$ stands for the $m \times n$ rectangle.

It is useful to introduce the parametrization for the Schur functions as the homomorphic map from the characteristic subalgebra Char(L) into the algebra $\mathbb{K}[\mu, \nu]$ of polynomials in commuting variables $\mu = \{\mu_i\}_{1\leqslant i\leqslant m}$ and $\nu = \{\nu_j\}_{1\leqslant j\leqslant n}$ (see [GPS2])

$$\frac{s_{[m|n]^k}(L)}{s_{[m|n]}(L)} \mapsto \frac{s_{[m|n]^k}(\mu, \nu)}{s_{[m|n]}(\mu, \nu)} = \sum_{1\leqslant i_1<\cdots<i_m\leqslant m} q^{-k} \mu_{i_1} \cdots \mu_{i_k} = e_k(q^{-1}\mu), \quad 1 \leqslant k \leqslant m, \tag{3.16}$$

$$\frac{s_{[m|n]_r}(L)}{s_{[m|n]}(L)} \mapsto \frac{s_{[m|n]_r}(\mu, \nu)}{s_{[m|n]}(\mu, \nu)} = \sum_{1\leqslant j_1<\cdots<j_r\leqslant n} (-q)^r \nu_{j_1} \cdots \nu_{j_r} = e_r(-q\nu), \quad 1 \leqslant r \leqslant n. \tag{3.17}$$

Here $e_k(\cdot)$ denotes the elementary symmetric polynomial in finitely many variables—the arguments of $e_k(\cdot)$.

Note that this parametrization is noncontradictory if we demand the Schur function $s_{[m|n]}(L)$ to be invertible and the ratios of the Schur functions in the left-hand side of (3.16) and (3.17) to be algebraically independent. Note that the indeterminates μ_i and ν_j can be treated as elements of an algebraic extension of the field of fractions of the central characteristic subalgebra[15].

With this parametrization the Cayley–Hamilton identity takes the form

$$(s_{[m|n]}(L))^2 \prod_{i=1}^{m} (L - \mu_i \,\mathrm{Id}) \cdot \prod_{j=1}^{n} (L - \nu_j \,\mathrm{Id}) \equiv 0.$$

Recall that since we are now dealing with the REA all matrix products have the usual sense. The above totally factorized form of the Cayley–Hamilton identity justifies an interpretation of the indeterminates $\{\mu_i\}$ and $\{\nu_j\}$ as, respectively, *'even'* and *'odd'* eigenvalues of the quantum super-matrix L.

Remark 8. As was shown in [GPS2], the map (3.16) and (3.17) allows us to parameterize all the elements of the characteristic subalgebra in terms of μ and ν. In particular,

$$s_{[m|n]}(L) \mapsto s_{[m|n]}(\mu, \nu) = \prod_{i=1}^{m} \prod_{j=1}^{n} (q^{-1}\mu_i - q\nu_j).$$

[15] As follows from the results of [GPS2], the characteristic subalgebra of REA considered as a ring is an integer domain and the field of fractions can be correctly defined.

Therefore, the invertibility of the Schur function $s_{[m|n]}(L)$ in terms of the spectrum of L is equivalent to invertibility of all factors $(q^{-1}\mu_i - q\nu_j)$.

Now we describe 'sl-reduction' of the REA and interrelations between this algebra and the RTT one. Let us rewrite the defining relations of the REA $\mathcal{L}(R_q)$ in the following form:

$$R_{12}L_1R_{12}L_1R_{12}^{-1} - L_1R_{12}L_1 = 0 \quad \text{or} \quad L_1R_{12}L_1R_{12}^{-1} - R_{12}^{-1}L_1R_{12}L_1 = 0. \tag{3.18}$$

Here as usual $L = \|L_i^j\|_{1\leqslant i,j\leqslant N}$ $(N = \dim V)$ and R is a skew-invertible Hecke symmetry of the bi-rank $(m|n)$.

One can easily prove the centrality of elements $p_k(L) = \mathrm{Tr}_R(L^k)$ which generates the characteristic subalgebra of the REA. Indeed, consecutively multiplying the defining relation (1.7) by the matrix L_1 from the left we get the following relations:

$$R_{12}L_1R_{12}L_1^k - L_1^kR_{12}L_1R_{12} = 0 \quad \Leftrightarrow \quad L_1R_{12}L_1^kR_{12}^{-1} - R_{12}^{-1}L_1^kR_{12}L_1 = 0, \qquad \forall k \in \mathbb{Z}_+.$$

Then, by applying the R-trace $\mathrm{Tr}_{R(2)}$ in the second space from the last relation and using the property

$$\mathrm{Tr}_{R(2)}\big(R_{12}^{\pm}X_1R_{12}^{\mp}\big) = \mathrm{Id}_1\,\mathrm{Tr}_R(X)$$

we find $p_k(L)L - Lp_k(L) = 0$. Emphasize once more that this proof uses only the skew invertibility of R and is valid for any $GL(m|n)$ type REA.

Now, apply a linear shift to generators of a $GL(m|n)$ type REA:

$$L_i^j \to L_i^j - \frac{\hbar}{q-q^{-1}}\delta_i^j 1_{\mathcal{L}} \tag{3.19}$$

with a nonzero numerical parameter $\hbar$ (we retain the same notations for the new generators and for the corresponding matrix $L = \|L_i^j\|$). Using the Hecke condition on R, for the new generators we get relations (1.10) of the mREA $\mathcal{L}(R_q, \hbar)$

$$R_{12}L_1R_{12}L_1 - L_1R_{12}L_1R_{12} = \hbar(R_{12}L_1 - L_1R_{12}).$$

Recall that the semiclassical counterpart of the standard mREA is a Poisson pencil considered in section 2.

If the bi-rank components $m \neq n$, then the R-trace of the unit matrix is nonzero in virtue of the following fomula (see [GPS3]):

$$\mathrm{Tr}_R(\mathrm{Id}) = \mathrm{Tr}(C) = q^{n-m}(m-n)_q, \qquad k_q = \frac{q^k - q^{-k}}{q-q^{-1}}.$$

In this case we are able to make an 'sl-reduction' of the mREA (or REA). To this end we pass from the set of generators $\{L_i^j\}$ to another set in which the central element $\ell = p_1(L) = \mathrm{Tr}_R(L)$ is chosen as one of the generators. Let us extract the R-traceless part of the matrix L:

$$F = L - \frac{\ell}{\mathrm{Tr}_R(\mathrm{Id})}\,\mathrm{Id}, \qquad \ell = \mathrm{Tr}_R(L). \tag{3.20}$$

Then the defining relations for the mREA can be rewritten in terms of generators F_i^j and ℓ as follows:

$$R_{12}F_1R_{12}F_1 - F_1R_{12}F_1R_{12} = \left(\hbar 1_{\mathcal{L}} - \frac{(q-q^{-1})}{\mathrm{Tr}_R(\mathrm{Id})}\ell\right)(R_{12}F_1 - F_1R_{12}), \qquad F\ell = \ell F. \tag{3.21}$$

Here the generators F_i^j are linearly dependent since $\mathrm{Tr}_R(F) = 0$.

We see that contrary to the classical case, the set of generators F_i^j does not generate a subalgebra in mREA. But the centrality of ℓ allows us to consider a quotient algebra

$$\mathcal{SL}(R_q, \hbar) = \mathcal{L}(R_q, \hbar)/\langle \ell \rangle. \tag{3.22}$$

The defining relations for the $\mathcal{SL}(R_q, \hbar)$ generators F_i^j (we keep the same notation for them) can be obtained from (3.21) by setting $\ell = 0$. The passage to the quotient $\mathcal{SL}(R_q, \hbar)$ (resp. $\mathcal{SL}(R_q)$) will be referred to as an sl-reduction of the mREA $\mathcal{L}(R_q, \hbar)$ (resp., $\mathcal{L}(R_q)$).

At last, we define *the truncated* mREA $\tilde{\mathcal{L}}(R_q, \hbar)$ as an associative algebra generated by the elements F_i^j and ℓ subject to the rules:

$$R_{12}F_1R_{12}F_1 - F_1R_{12}F_1R_{12} = \hbar(R_{12}F_1 - F_1R_{12}), \qquad F\ell = \ell F, \qquad \mathrm{Tr}_R(F) = 0.$$

We complete this section by discussing a covariance property of the REA with respect to the adjoint coaction of the RTT algebra. The coaction is based on the Hopf algebra structure which can be defined in the RTT algebra under the additional condition that the element $\det_q T$ is central (see section 1). This condition is necessary for definition of the antipodal map. Note that in the standard case this coaction allows us to endow the REA with a covariant $U_q(sl(N))$-module structure since the quantum group is restricted dual to the standard RTT algebra quotiented by the condition $\det_q T = 1$.

The coproduct in the RTT algebra reads $\Delta(T) = T \dot{\otimes} T$ [FRT]. Hereafter, the notation $A \dot{\otimes} B$ means the matrix product of two equal size matrices A and B where their matrix elements are not multiplied but tensorized. Thus, matrix elements of the matrix $A \dot{\otimes} B$ are

$$(A \dot{\otimes} B)_i^j = A_i^k \otimes B_k^j.$$

Consequently, all the algebras considered above ($\mathcal{L}(R_q), \mathcal{L}(R_q, \hbar), \tilde{\mathcal{L}}(R_q, \hbar)$ and $\mathcal{SL}(R_q, \hbar)$) can be given a structure of the (right) adjoint comodule over the RTT algebra. The adjoint coaction δ_r on the generators can be presented as follows:

$$\delta_r\big(A_i^j\big) = \sum_{k,p=1}^{N} A_k^p \otimes T_i^k S\big(T_p^j\big) \qquad \text{or} \qquad \delta_r(A) = TAS(T),$$

where A stands for L or F and $S(\cdot)$ denotes the antipodal map. The products in the above algebras are covariant with respect to this coaction, i.e., the multiplication in these algebras commutes with the coaction.

Thus, in the standard case all these algebras can be endowed with an action of the QG $U_q(sl(n))$ so that their products are coordinated with this action in the sense of formula (1.8).

4. Elements of the mREA representation theory

In this section, we reproduce some elements of representation theory of the REA related to a skew-invertible Hecke symmetry of general type and compare our approach with that arising from [FRT] and [LS] which is valid for the standard REA.

Our method of constructing the representation theory of the REA is based on treating the space $\mathcal{L} = \mathrm{span}\big(L_i^j\big)$ as a space of endomorphisms of the basic (fundamental) space V. Indeed, as was shown in [GPS3], the dual spaces V and V^* possess the basis sets $\{x_i\}$ and $\{x^j\}$ in which the right and left pairings are uniquely (up to a nonzero factor) defined by the formulae

$$V \otimes V^* \to \mathbb{K}: \quad x_i \otimes x^j \mapsto \delta_i^j \qquad \text{and} \qquad V^* \otimes V \to \mathbb{K}: \quad x^j \otimes x_i \mapsto B_i^j, \tag{4.1}$$

where $\|B_i^j\|$ is the matrix corresponding to the operator (3.4) in the basis $\{x_i\}$. Recall that this matrix is invertible for any skew-invertible Hecke symmetry. In order to construct the Schur–Weyl category of the REA representations we need only the skew-invertibility of a Hecke symmetry.

The crucial point is that the map $\rho_V : \mathcal{L} \to \mathrm{End}(V)$ defined by the rule

$$\rho_V\big(L_i^j\big) \triangleright x_k = x_i B_k^j \tag{4.2}$$

realizes an irreducible representation of the algebra $\mathcal{L}(R_q, 1)$ (the symbol $\triangleright$ stands for the action of a linear operator). This action together with the pairing $\langle x^j, x_i\rangle = B_i^j$ is the motivation for treatment of L_i^j as the element $x_i \otimes x^j$.

If R is the usual flip (therefore $B_i^j = \delta_i^j$) we get the basic (fundamental) representation of the algebra $U(gl(n))$. If R is a super-flip then B is the parity operator: $B(x) = (-1)^{\bar{x}}x$ where x is a homogeneous element and $\bar{x}$ is its parity.

Let us consider an example, namely the REA related to the $U_q(sl(2))$ Hecke symmetry. In the basis $\{x_i \otimes x_j\} \in V^{\otimes 2}$ we get the following matrix:

$$R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}.$$

Then the matrix B becomes $\mathrm{diag}(q^{-1}, q^{-3})$. Let $L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the matrix of the mREA generators. Explicitly, the corresponding mREA $\mathcal{L}(R_q, \hbar)$ is defined by the following system:

$$\begin{aligned} &qab - q^{-1}ba = \hbar b, && q(bc - cb) - (q - q^{-1})a(d - a) = \hbar(a - d), \\ &qca - q^{-1}ac = \hbar c, && q(cd - dc) - (q - q^{-1})ca = -\hbar c, \\ &ad - da = 0, && q(db - bd) - (q - q^{-1})ab = -\hbar b. \end{aligned} \tag{4.3}$$

We present this system in terms of another set of generators $\{\ell, h, b, c\}$, where $\ell = q^2\,\mathrm{Tr}_R L = q^{-1}a + q^1 d$ and $h = a - d$. The multiplication rules (4.3) take the form

$$\begin{aligned} &q^2hb - bh + (q - q^{-1})\ell b = 2_q\hbar b, && b\ell - \ell b = 0, \\ &q^2ch - hc + (q - q^{-1})\ell c = 2_q\hbar c, && c\ell - \ell c = 0, \\ &2_q q(bc - cb) + (q^2 - 1)h^2 + (q - q^{-1})\ell h = 2_q\hbar h, && h\ell - \ell h = 0. \end{aligned} \tag{4.4}$$

Observe that the element ℓ is central in this algebra. In what follows we also consider the quotient $\mathcal{SL}(R_q, \hbar) = \mathcal{L}(R_q, \hbar)/\langle\ell\rangle$. Thus, this algebra is generated by three elements b, h, c subject to the system

$$q^2hb - bh = 2_q\hbar b, \qquad q^2ch - hc = 2_q\hbar c, \qquad 2_q q(bc - cb) + (q^2 - 1)h^2 = 2_q\hbar h. \tag{4.5}$$

Note that at $q = 1$ this algebra $\mathcal{SL}(R_q, \hbar)$ coincides with $U(sl(2)_\hbar)$.

The explicit form of the representation (4.2) reads

$$\rho_V(a) = \begin{pmatrix} q^{-1} & 0 \\ 0 & 0 \end{pmatrix}, \qquad \rho_V(b) = \begin{pmatrix} 0 & q^{-3} \\ 0 & 0 \end{pmatrix},$$

$$\rho_V(c) = \begin{pmatrix} 0 & 0 \\ q^{-1} & 0 \end{pmatrix}, \qquad \rho_V(d) = \begin{pmatrix} 0 & 0 \\ 0 & q^{-3} \end{pmatrix}.$$

In order to get the corresponding representation of the sl-reduced algebra (4.5) we use the general recipe suggested in [S]. Let $\rho_U : \mathcal{L}(R_q, 1) \to \mathrm{End}(U)$ be an mREA representation and the central element $\mathrm{Tr}_R(L)$ is represented by a scalar operator

$$\rho_U(\mathrm{Tr}_R(L)) = \chi_1\,\mathrm{Id}_U,$$

where $\chi_1 = \chi(\mathrm{Tr}_R(L))$ is the value of a character $\chi : Z(\mathcal{L}(R_q, 1)) \to \mathbb{K}$ of the center $Z(\mathcal{L}(R_q, 1))$.

Then the straightforward calculation shows that the $\mathcal{SL}(R_q, 1)$ generators F_i^j are represented in the space $\mathrm{End}(U)$ by the following linear operators:

$$\bar{\rho}_U\big(F_i^j\big) = \frac{1}{\omega}\left(\rho_U\big(L_i^j\big) - \delta_i^j\frac{\chi_1}{\mathrm{Tr}\,C}\mathrm{Id}_U\right), \qquad \omega = 1 - (q - q^{-1})\frac{\chi_1}{\mathrm{Tr}\,C}. \tag{4.6}$$

Applying formula (4.6) to the above representation ρ_V we get the representation of the algebra $\mathcal{SL}(R_q, 1)$ (4.5)

$$\bar{\rho}_V(h) = w\begin{pmatrix} q & 0 \\ 0 & -q^{-1} \end{pmatrix}, \qquad \bar{\rho}_V(b) = w\begin{pmatrix} 0 & q^{-1} \\ 0 & 0 \end{pmatrix},$$
$$\bar{\rho}_V(c) = w\begin{pmatrix} 0 & 0 \\ q & 0 \end{pmatrix}, \qquad w = \frac{q^2+1}{q^4+1}. \tag{4.7}$$

Let us go back to a general case and introduce another basic (contragradient) representation $\rho_{V^*} : \mathcal{L}(R_q, 1) \to \mathrm{End}(V^*)$ of the algebra $\mathcal{L}(R_q, 1)$ defined by the formula

$$\rho_{V^*}\big(L_i^j\big) \triangleright x^k = -x^r R_{ri}^{kj}. \tag{4.8}$$

Note that in the classical case (when R is the usual flip) we have $\rho_{V^*}(a) = -\rho_V(a)^*$. Assuming R to be involutive and introducing an involution via R as discussed in the following section we can get such a contragradient representation. However, if R is not involutive this method fails.

In the case in question this representation is

$$\rho_{V^*}(a) = \begin{pmatrix} -q & 0 \\ 0 & 0 \end{pmatrix}, \qquad \rho_{V^*}(b) = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix},$$
$$\rho_{V^*}(c) = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \qquad \rho_{V^*}(d) = \begin{pmatrix} q^{-1}-q & 0 \\ 0 & -q \end{pmatrix}.$$

As was mentioned above, in [GPS3] there was constructed a rigid quasitensor Schur–Weyl category $SW(V)$ of vector spaces generated by V and V^* such that each object of this category can be endowed with an $\mathcal{L}(R_q, 1)$-module structure. The term 'rigid' means that with any object U this category contains its dual. Recall that an object U^* is dual to U if there exists nontrivial pairings $U \otimes U^* \to \mathbb{K}$ and $U^* \otimes U \to \mathbb{K}$ which are the categorical morphisms. A peculiarity of the case related to an even R is that (by assuming $\det_q T$ to be central, see section 1) the dual space to V can be identified with a subspace in some tensor power of the space V while for the general case the dual space V^* must be introduced independently.

Using the pairing (4.1) we introduce an important morphism $\mathrm{tr}_R : \mathcal{L} \to \mathbb{K}$ which will be referred to as a categorical trace

$$\mathrm{tr}_R\big(L_i^j\big) = \delta_i^j. \tag{4.9}$$

Remark 9. The fact that the above definition oftr$_R$ gives a categorical morphism follows from the identification of L_i^j and $x_i \otimes x^j$. Now the categorical trace just equals the pairing

$$\mathrm{tr}_R\big(L_i^j\big) = \langle x_i, x^j\rangle$$

which is a morphism (see [GPS3]).

Anyway, in a general case the initial skew-invertible braiding R can be extended in a unique way to the braidings $R_{U,W}$ for any two objects U and W of the Schur–Weyl category. Besides, all morphisms of the category are assumed to be natural or functorial as defined in [T]. This means that given two morphisms $f : U \to U'$ and $g : W \to W'$ the relation

$$(g \otimes f) \circ R_{U,W} = R_{U',W'} \circ (f \otimes g)$$

(where the symbol $\circ$ denotes the composition of the maps) is assumed to be fulfilled. By putting $W' = W$ and $g = \mathrm{Id}$ we get a condition on a morphism f which means that f is covariant.

As we said above the space $\mathcal{L}$ is identified with the product $V \otimes V^*$. Consequently, this space is an object of the Schur–Weyl category.

Definition 10. *Given an object U of the Schur–Weyl category, representations $\rho_U : \mathcal{L}(R_q, 1) \to \mathrm{End}(U)$ is called R-invariant or equivariant if its restriction to the space $\mathcal{L}$ is a categorical morphism.*

For example, the above representations ρ_V and ρ_{V^*} are equivariant.

Our next goal is to exhibit a way of multiplying equivariant representations of the algebra $\mathcal{L}(R_q, 1)$. To this end we need the braiding

$$R_{\mathrm{End}} : \mathrm{End}(V)^{\otimes 2} \to \mathrm{End}(V)^{\otimes 2},$$

which is a particular but very important case of the aforementioned braidings $R_{U,W}$.

Its explicit form in the basis $\{L_1 \dot{\otimes} L_2\}$ of the space $\mathcal{L}^{\otimes 2}$ reads as follows [GPS3]:

$$R_{\mathrm{End}}(L_1 \dot{\otimes} L_2) = \mathrm{Tr}_{(0)}\big(R_{10} L_1 R_{10}^{-1} \dot{\otimes} L_1 R_{10} \Psi_{02}\big) P_{12}, \tag{4.10}$$

where the operator Ψ is defined in (3.2). (The notation $\dot{\otimes}$ was introduced in the previous section. For instance, the entries of the matrix $L_1 \dot{\otimes} L_2$ where $L_1 = L \otimes \mathrm{Id}$, $L_2 = \mathrm{Id} \otimes L$ are $L_i^j \otimes L_k^l$.)

In the tensor power $\mathcal{L}^{\otimes k}$ there is a basis different from that $\big\{L_{i_1}^{j_1} \otimes L_{i_2}^{j_2} \otimes \cdots \otimes L_{i_k}^{j_k}\big\}$. This new basis is composed from the matrix elements of the tensor product $L_{\bar{1}} \dot{\otimes} L_{\bar{2}} \dot{\otimes} \cdots \dot{\otimes} L_{\bar{k}}$ and it turns out to be more convenient for our calculations. Here in accordance with the general definition (3.8) we set

$$L_{\bar{1}} = L \otimes \mathrm{Id}^{\otimes(k-1)}, \qquad L_{\overline{p+1}} = R_p L_{\bar{p}} R_p^{-1}, \qquad p \leqslant k-1. \tag{4.11}$$

Using the skew-inverse operator Ψ one can prove that the matrix elements of $L_{\bar{1}} \dot{\otimes} L_{\bar{2}} \dot{\otimes} \cdots \dot{\otimes} L_{\bar{k}}$ are in one-to-one correspondence with those of $L_1 \dot{\otimes} L_2 \dot{\otimes} \cdots \dot{\otimes} L_k$ and thus form the basis in $\mathcal{L}^{\otimes k} \cong (\mathrm{End}(V))^{\otimes k}$.

Making use of (4.10) we find that

$$R_{\mathrm{End}}(L_{\bar{1}} \dot{\otimes} L_{\bar{2}}) = L_{\bar{2}} \dot{\otimes} L_{\bar{1}}, \tag{4.12}$$

and, as a consequence,

$$R_{\mathrm{End}}(L_{\bar{k}} \dot{\otimes} L_{\bar{p}}) = L_{\bar{p}} \dot{\otimes} L_{\bar{k}}, \qquad \forall p > k, \qquad k, p \in \mathbb{Z}_+.$$

Note that each homogeneous component of the associated graded algebra $\mathrm{Gr}\mathcal{L}(R_q, \hbar)$ can be identified with an object of the category $SW(V)$. So, the braiding R_{End} can be extended onto the whole algebra $\mathrm{Gr}\mathcal{L}(R_q, \hbar)$ and, consequently, to the algebra $\mathcal{L}(R_q, \hbar)$ (in particular, to $\mathcal{L}(R_q, 1)$). Besides, the braidings transposing elements of $\mathcal{L}(R_q, \hbar)$ and elements of arbitrary objects of the category $SW(V)$ are also defined (see [GPS3] for detail). Below the symbol $\mathsf{R}(a \otimes b)$ stands for the general form of braiding transposing a and b which are elements of objects of the category $SW(V)$ (in particular, one or both of them can belong to $\mathcal{L}(R_q, \hbar)$).

Our next goal is to introduce a coproduct which endows $\mathcal{L}(R_q, 1)$ with a braided bi-algebra structure. This enables us to multiply its equivariant representations. In order to describe the coproduct and the corresponding braided bi-algebra structure we need the following definition.

Definition 11. *Given any skew-invertible Hecke symmetry R, introduce a braided associative algebra $\mathbf{L}(R_q)$ by the data:*

(1) As a vector space over the field $\mathbb{K}$ the algebra $\mathbf{L}(R_q)$ is isomorphic to the tensor product of two copies of mREA $\mathcal{L}(R_q, 1)$

$$\mathbf{L}(R_q) \cong \mathcal{L}(R_q, 1) \otimes \mathcal{L}(R_q, 1).$$

(2) The product $*: \mathbf{L}(R_q)^{\otimes 2} \to \mathbf{L}(R_q)$ *is defined by the rule*

$$(a_1 \otimes b_1) * (a_2 \otimes b_2) = a_1 a_2' \otimes b_1' b_2, \qquad \forall a_i \otimes b_i \in \mathbf{L}(R_q), \tag{4.13}$$

where $a_1 a_2'$ *and* $b_1 b_2'$ *are the products of elements of mREA and*

$$a_2' \otimes b_1' = \mathsf{R}(b_1 \otimes a_2). \tag{4.14}$$

Note that the associativity of the $*$-product is proved in [GPS3].

Let us define a coproduct $\Delta : \mathcal{L}(R_q, 1) \to \mathbf{L}(R_q)$ as a linear map of the following form:

$$\begin{aligned} &\Delta(e_{\mathcal{L}}) = e_{\mathcal{L}} \otimes e_{\mathcal{L}}, \\ &\Delta\big(L_i^j\big) = L_i^j \otimes e_{\mathcal{L}} + e_{\mathcal{L}} \otimes L_i^j - (q - q^{-1}) \textstyle\sum_k L_i^k \otimes L_k^j, \\ &\Delta(ab) = \Delta(a) * \Delta(b) \qquad \forall a, b \in \mathcal{L}(R_q, 1). \end{aligned} \tag{4.15}$$

In addition to (4.15), we introduce a linear map $\varepsilon : \mathcal{L}(R_q, 1) \to \mathbb{K}$

$$\varepsilon(e_{\mathcal{L}}) = 1, \qquad \varepsilon\big(L_i^j\big) = 0, \qquad \varepsilon(ab) = \varepsilon(a)\varepsilon(b) \quad \forall a, b \in \mathcal{L}(R_q, 1). \tag{4.16}$$

Proposition 12 ([GPS3]). *The maps* Δ *and* ε *endow the algebra* $\mathcal{L}(R_q, 1)$ *with a bi-algebra structure. This means that the relations*

$$(\mathrm{Id} \otimes \Delta)\Delta = (\Delta \otimes \mathrm{Id})\Delta, \qquad (\mathrm{Id} \otimes \varepsilon)\Delta = \mathrm{Id} = (\varepsilon \otimes \mathrm{Id})\Delta$$

hold and moreover Δ *is a homomorphism of the algebra* $\mathcal{L}(R_q, 1)$ *into* $\mathbf{L}(R_q)$.

Remark 13. The above braided bialgebra structure was deduced from the braided bialgebra structure in $\mathcal{L}(R_q)$ discovered by Majid (see [M3, M4]). A passage from the latter structure to the former one can be given by a shift of generators (3.19). Nevertheless, namely, in the form (4.15) the coproduct involved in this construction is very useful for defining 'braided vector fields' (see section 8). Also, note that for $q = 1$ this coproduct takes the well-known additive form on generators of a Lie algebra or its generalized analog discussed in the following section.

Now, we are able to define a product of two representations of the algebra $\mathcal{L}(R_q, 1)$.

Proposition 14 [GPS3]. *Given two equivariant mREA modules U and W, let* $\rho_U : \mathcal{L}(R_q, 1) \to \mathrm{End}(U)$ *and* $\rho_W : \mathcal{L}(R_q, 1) \to \mathrm{End}(W)$ *be the corresponding equivariant representations. Consider a map* $\rho_{U\otimes W} : \mathbf{L}(R_q) \to \mathrm{End}(U \otimes W)$ *defined by the following rule:*

$$\begin{aligned} &\rho_{U\otimes W}(a \otimes b) \triangleright (u \otimes w) = (\rho_U(a) \triangleright u') \otimes (\rho_W(b') \triangleright w), \\ &\forall a \otimes b \in \mathbf{L}(R_q), \qquad \forall u \in U, \quad \forall w \in W, \end{aligned} \tag{4.17}$$

where

$$u' \otimes b' = \mathsf{R}(b \otimes u).$$

Then the map (4.17) defines a representation of the braided algebra $\mathbf{L}(R_q)$ *in the space* $U \otimes W$.

This proposition implies the following corollary.

Corollary 15. *Let U and W be two* $\mathcal{L}(R_q, 1)$*-modules with equivariant representations* ρ_U *and* ρ_W*. Then the map* $\mathcal{L}(R_q, 1) \to \mathrm{End}(U \otimes W)$ *given by*

$$a \mapsto \rho_{U\otimes W}(\Delta(a)), \qquad \forall a \in \mathcal{L}(R_q, 1), \tag{4.18}$$

where the coproduct Δ *and the map* $\rho_{U\otimes W}$ *are given respectively by formulae (4.15) and (4.17), is an equivariant representation of* $\mathcal{L}(R_q, 1)$.

Using this result we can endow any tensor products of V and V^* (as well as subspaces of these tensor products, extracted by Young projectors) with a $\mathcal{L}(R_q, 1)$-module structure so that the corresponding representation is equivariant (see [GPS3]).

As an important example we consider an analog of the 'adjoint' representation $\rho_{V\otimes V^*}$, arising from those ρ_V and ρ_{V^*}. To this end, we need an explicit form of the braiding $R_{\mathrm{End},V} : \mathrm{End}(V) \otimes V \to V \otimes \mathrm{End}(V)$ since it comes in construction of the product of representations.

The matrix of this braiding turns out to be [GPS3]:

$$R_{\mathrm{End},V}(L_1 \dot{\otimes} x_2) = x_2 \dot{\otimes} \mathrm{Tr}_{(0)}(R_{10}L_0\Psi_{02})P_{12}. \tag{4.19}$$

The action

$$\rho_{V\otimes V^*}\big(L_{i_1}^{j_1}\big) \triangleright L_{i_2}^{j_2} = (\rho_V \otimes \rho_{V^*}) \circ \big(\Delta\big(L_{i_1}^{j_1}\big) \otimes (x_{i_2} \otimes x^{j_2})\big)$$

or, in matrix form, $\rho_{V\otimes V^*}(L_1) \triangleright L_2$ can be easily calculated with the use of formulae (4.2), (4.8), (4.15), and (4.19). The result is given by a somewhat complicated expression:

$$\rho_{V\otimes V^*}(L_1) \triangleright L_2 = (q - q^{-1})L_1 I_2 + L_1 B_2 P_{12} - \mathrm{Tr}_{(0)}(R_{10}L_1R_{10}\Psi_{02})P_{12}.$$

Nevertheless, in the basis of quantum matrix copies (4.11) the adjoint action has a nice and simple form, namely

$$\rho_{V\otimes V^*}(L_{\bar{1}}) \triangleright L_{\bar{2}} = L_1 R_{12} - R_{12} L_1. \tag{4.20}$$

This example shows again that basis of copies (4.11) is a very useful tool.

Now, we restrict ourselves to the standard case and explain a way of constructing a family of representations of the corresponding (m)REA arising from methods [FRT] and [LS]. Note that the approach of [LS] was used in [DS] for constructing representation theory of the braided Lie algebra $sl(2)_q$ (see section 5).

The authors of [FRT] suggested a way of finding the center of the QG $U_q(\mathfrak{g})$. This center can be easily calculated via the formula $\mathrm{Tr}_R L^k$, where L is a matrix with entries belonging to the QG. Lately, it was understood that this matrix is subject to the defining relations of the corresponding REA. The matrix L was defined in [FRT] via the formula

$$L = S(L^-)L^+, \tag{4.21}$$

where $L^\pm$ are the matrices composed of the quantum group generators and $S(\cdot)$ is the antipode.

Let us consider the simplest example related to the QG $U_q(sl(2))$. In this case with the use of the basis $\{\ell, h, b, c\}$ discussed at the beginning of this section we get the following parametrization of the mREA generators via the generators E, F and H of the QG $U_q(sl(2))$:

$$\begin{aligned} &h = -\frac{\lambda}{q}(qEF - q^{-1}FE), \qquad && b = \lambda q^{-H}E, \\ &c = \lambda q^{-H-2}F, && \ell = q^{2H-1} + q^{-2H-3} + \frac{\lambda^2}{q^2}FE, \end{aligned} \tag{4.22}$$

where we set for shortness $\lambda = q - q^{-1}$. The generators E, F and H are normalized as follows:

$$q^H E = qEq^H, \qquad q^H F = q^{-1}Fq^H, \qquad [E, F] = \frac{q^{2H} - q^{-2H}}{q - q^{-1}}.$$

However, the generators ℓ, h, b, c are not independent in the above parametrization. In [FRT] the problem of finding relations between these generators was not considered. Such relations in the $U_q(sl(n))$ case were found in [LS]. In that paper for the QG $U_q(sl(n))$ a family of generators similar to those above was constructed. In the simplest case ($n = 2$), these generators X_0, $X_\pm$ and C satisfy the system of relations [LS]

$$\begin{aligned} &q^2X_0X_+ - X_+X_0 = qCX_+,\\ &q^{-2}X_0X_- - X_-X_0 = -q^{-1}CX_-,\\ &X_+X_- - X_-X_+ + (q^2 - q^{-2})X_0^2 = (q + q^{-1})CX_0,\\ &CX_m = X_mC \quad (m = 0, \pm) \end{aligned} \tag{4.23}$$

and

$$C^2 - (q - q^{-1})^2\left(X_0^2 + \frac{qX_-X_+ + q^{-1}X_+X_-}{q + q^{-1}}\right) = 1. \tag{4.24}$$

It is easy to see that upon setting

$$X_0 = \frac{qC}{2_q}h, \qquad X_+ = qCb, \qquad X_- = qCc$$

the first three lines of (4.23) transform precisely into relations (4.5) with $\hbar = 1$. Thus, the algebra $\mathcal{SL}(R_q, 1)$ is embedded in the QG $U_q(sl(2))$ extended by the element C^{-1}.

Since C is central, it becomes a nontrivial scalar in each finite-dimensional representation of the QG $U_q(sl(2))$. This enables one to get a representation of the algebra $\mathcal{SL}(R_q, \hbar)$ once a $U_q(sl(2))$ representation is given. Note that such a representation of the algebra $\mathcal{SL}(R_q, \hbar)$ can be treated as that of $\mathcal{L}(R_q, \hbar)$ but with the image of ℓ depending on the images of b, c and h (due to (4.24)). Thus, we can get a subfamily of all finite-dimensional representations of the corresponding REA constructed by the general method. The same is valid for all QG $U_q(sl(n))$.

In conclusion, we want to emphasize that our method of constructing the REA representations does not give rise to non-equivariant ones. For example, the standard REA has one-dimensional representations, which cannot be obtained either by our method or by that of [LS]. We refer the reader to [K, Mu1] for such type representations. Also note that our approach is valid for a generic q whereas the method of [LS] allows us to consider the case of special q (roots of unity). This case was considered in [DS]. Besides, this method enables one to construct Verma type modules for the standard algebra $\mathcal{SL}(R_q, \hbar)$ (also, see [LS]). Note that in general we cannot define such type modules.

However, by comparing our approach and that from [LS] we would like to stress again that the latter one cannot be applied to the REA connected with nonstandard braidings since the corresponding QG like objects are not known for the general type braidings.

5. Braided Lie algebras

In this section, we discuss the role of the mREA in the definition of the quantum (braided) analogs of a Lie algebra. We are mainly interested in the Lie algebra type objects similar to $gl(n)$ and $sl(n)$. For such objects the problem can be formulated as follows. Let $R : V^{\otimes 2} \to V^{\otimes 2}$ be a skew-invertible braiding. Consider the space $\mathrm{End}(V)$ equipped with the braiding $R_{\mathrm{End}} : \mathrm{End}(V)^{\otimes 2} \to \mathrm{End}(V)^{\otimes 2}$ (4.12). We want to introduce a braided analog of the Lie bracket in the space $\mathrm{End}(V)$, which is an R_{End}-invariant operator

$$[\,,\,]: \quad \mathrm{End}(V)^{\otimes 2} \to \mathrm{End}(V)$$

and to define the corresponding *enveloping algebra* $U(\mathrm{End}(V))$ with the good deformation property. Actually, we first introduce an analog of the enveloping algebra $U(gl(n))$ and then define a $gl(n)$ type bracket.

Before going into detail we would like to make some historical comments. The first generalization of a Lie algebra was a super-Lie algebra. We were informed by Gerstenhaber and Stasheff, to whom we express our profound gratitude, that a super-version of the Jacobi identity was introduced in [N] (also, in the 1940–1950s super-algebra type objects were popular in connection with the Whitehead product). Lately, the super-algebras were intensively studied by physicists in the frameworks of models with fermion–boson symmetries.

The basic example is a super-Lie algebra $gl(m|n)$ defined in the space $\mathrm{End}(V)$ where $V = V_{(m|n)} = V_{\bar{0}} \oplus V_{\bar{1}}$ is a $\mathbb{Z}_2$ graded space and $\dim V_{\bar{0}} = m$, $\dim V_{\bar{1}} = n$. A braiding R is a super-flip $R(x \otimes y) = (-1)^{\bar{x}\bar{y}} y \otimes x$.

Attempts to introduce the Lie algebra type objects graded by other finite commutative group were undertaken in the late 1970s [Sch].

The next generalization of the Lie algebra notion was related to the involutive symmetries $R : R^2 = \mathrm{Id}$. In [G1] the following notion was introduced.

Definition 16. *The data*

$$(V, R : V^{\otimes 2} \to V^{\otimes 2}, [\,,\,] : V^{\otimes 2} \to V)$$

where R is an involutive symmetry are called a generalized Lie algebra if the axioms below hold true:

(1) $[\,,\,]R(X \otimes Y) = -[X, Y]$*;*
(2) $[\,,\,][\,,\,]_{12}(I + R_{12}R_{23} + R_{23}R_{12})(X \otimes Y \otimes Z) = 0$*;*
(3) $R[\,,\,]_{23}(X \otimes Y \otimes Z) = [\,,\,]_{12}R_{23}R_{12}(X \otimes Y \otimes Z)$

for all $X, Y, Z \in V$.

Note that the generalized Jacobi identity (axiom 2) can also be presented in one of the following equivalent forms:

2.a. $[\,,\,][\,,\,]_{23}(I + R_{12}R_{23} + R_{23}R_{12})(X \otimes Y \otimes Z) = 0$;
2.b. $[\,,\,][\,,\,]_{12}(X \otimes (Y \otimes Z - R(Y \otimes Z))) = [X, [Y, Z]]$;
2.c. $[\,,\,][\,,\,]_{23}((X \otimes Y - R(X \otimes Y)) \otimes Z) = [[X, Y], Z]$.

Let us denote the generalized Lie algebra originated from a vector space V and a braiding R as $\mathfrak{g}(V, R)$. Its enveloping algebra $U(\mathfrak{g}(V, R))$ can be naturally defined as a quotient of the free tensor algebra $T(\mathfrak{g}(V, R))$:

$$U(\mathfrak{g}(V, R)) = T(\mathfrak{g}(V, R))/\langle X \otimes Y - R(X \otimes Y) - [X, Y]\rangle \quad \forall X, Y \in V.$$

The enveloping algebra $U(\mathfrak{g}(V, R))$ has the good deformation property. In particular, due to the PBW-like theorem [PP] the associated graded algebra $\mathrm{Gr}U(\mathfrak{g}(V, R))$ is canonically isomorphic to the algebra

$$\mathrm{Sym}(\mathfrak{g}(V, R)) = T(\mathfrak{g}(V, R))/\langle \mathrm{Im}(\mathrm{Id} - R)\rangle.$$

Besides, it becomes a *braided Hopf algebra*, being equipped with an appropriate coproduct Δ and an antipode S. These operators have the classical form $\Delta(X) = X \otimes 1 + 1 \otimes X$ and $S(X) = -X$ on elements $X \in V$ and can be naturally extended to the whole algebra with the use of the operator R (see [G2]).

The axioms of definition 16 are satisfied by the following data:

$$(\mathrm{End}(V), R_{\mathrm{End}} : \mathrm{End}(V)^{\otimes 2} \to \mathrm{End}(V)^{\otimes 2}, [\,,\,] : \mathrm{End}(V)^{\otimes 2} \to \mathrm{End}(V)), \tag{5.1}$$

where R_{End} is an extension of a skew-invertible involutive symmetry $R : V^{\otimes 2} \to V^{\otimes 2}$ to the space $\mathrm{End}(V)^{\otimes 2}$ (see section 4) and the bracket is defined by

$$[\,,\,] = \circ(\mathrm{Id} - R_{\mathrm{End}}). \tag{5.2}$$

Here $\circ : \mathrm{End}(V)^{\otimes 2} \to \mathrm{End}(V)$ is the usual product in the algebra $\mathrm{End}(V)$. For the generalized Lie algebra (5.1) we use the notation $gl(V, R)$.

Now, we suppose $R = R_q$ to be a Hecke symmetry. Consider the corresponding mREA $\mathcal{L}(R_q, 1)$. As was shown in [GPS3], the algebra $\mathcal{L}(R_q, 1)$ possesses the good deformation property.

Now we define a *braided Lie algebra* such that its enveloping algebra coincides with the mREA $\mathcal{L}(R_q, 1)$. To this end we rewrite the mREA multiplication rules (1.10) in the following form:

$$L_{\bar 1} L_{\bar 2} - R_{12}^{-1} L_{\bar 1} L_{\bar 2} R_{12} = L_1 R_{12} - R_{12} L_1, \tag{5.3}$$

where the matrices $L_{\bar k}$ are defined in (4.11). Then, on the linear space $\mathcal{L} = \mathrm{span}\big(L_i^j\big) \cong \mathrm{End}(V)$ we introduce a Lie type bracket $[\,,\,] : \mathcal{L}^{\otimes 2} \to \mathcal{L}$ setting by definition

$$[L_{\bar 1}, L_{\bar 2}] = L_1 R_{12} - R_{12} L_1. \tag{5.4}$$

This bracket can be written in a form, similar to (5.2). For this purpose we define the linear operator $Q : \mathcal{L}^{\otimes 2} \to \mathcal{L}^{\otimes 2}$:

$$Q(L_{\bar 1} \dot\otimes L_{\bar 2}) = R_{12}^{-1} L_{\bar 1} \dot\otimes L_{\bar 2} R_{12}. \tag{5.5}$$

Then, as can be easily seen from (5.3), the bracket (5.4) reads

$$[\,,\,] = \circ(\mathrm{Id} - Q) \tag{5.6}$$

where $\circ$ has the same meaning as above. Note that the multiplication of the operators L_i^j can be calculated on the basis of (4.2)

$$L_{i_1}^{j_1} \circ L_{i_2}^{j_2} = B_{i_2}^{j_1} L_{i_1}^{j_2}. \tag{5.7}$$

One of the important distinction between the bracket (5.4) and that (5.2) of the generalized Lie algebra $gl(V, R)$ consists in the fact that the operator Q differs from the operator R_{End} whereas if R is an involutive symmetry these operators are equal to each other.

In terms of the operator Q we can express an operator $\mathcal{S}_q : \mathcal{L}^{\otimes 2} \to \mathcal{L}^{\otimes 2}$ possessing the property

$$\mathcal{L}^{(2)}(R_q) = \mathrm{Im}\, \mathcal{S}_q,$$

where $\mathcal{L}^{(2)}(R_q) \subset \mathcal{L}^{\otimes 2}$ is the second-order homogeneous component of the REA $\mathcal{L}(R_q)$. Such an operator $\mathcal{S}_q$ can be defined by the following formula (see [GPS3]):

$$\mathcal{S}_q = \frac{1}{2_q^2}((q^2 + q^{-2})\,\mathrm{Id} + Q + Q^{-1}).$$

Thus, the operator $\mathcal{S}_q$ can be treated as a total q-symmetrizer on the space $\mathcal{L}^{\otimes 2}$.

Theorem 17. *The bracket (5.4) possesses the following properties:*

(1) The bracket is skew-symmetric in the following sense:

$$[\,,\,]\mathcal{S}_q(L_{\bar 1} \dot\otimes L_{\bar 2}) = 0, \tag{5.8}$$

where $\mathcal{S}_q$ is the symmetrizer introduced above.

(2) The bracket satisfies the generalized Jacobi identity of the form

$$[\,,\,][\,,\,]_{23}((X \otimes Y - Q(X \otimes Y)) \otimes Z) = [[X, Y], Z] \qquad \forall X, Y, Z \in \mathcal{L},$$

otherwise stated, the adjoint action defined by

$$\mathrm{ad}\, L_i^j\big(L_k^l\big) = \big[L_i^j, L_k^l\big] \tag{5.9}$$

is a representation of the algebra $\mathcal{L}(R_q, 1)$.

(3) The bracket is R_{End}-invariant. This means that

$$\begin{aligned} R_{\mathrm{End}}[\,,\,]_{23} &= [\,,\,]_{12}(R_{\mathrm{End}})_{23}(R_{\mathrm{End}})_{12}, \\ R_{\mathrm{End}}[\,,\,]_{12} &= [\,,\,]_{23}(R_{\mathrm{End}})_{12}(R_{\mathrm{End}})_{23}, \end{aligned} \tag{5.10}$$

where both sides of the above equalities are operators acting in $\mathcal{L}^{\otimes 3}$.

Proof. To prove the first statement of the theorem, we note that the operator Q has the minimal polynomial of the form [GPS3]

$$(Q + q^2\mathrm{Id})(Q + q^{-2}\mathrm{Id})(Q - \mathrm{Id}) = 0$$

which in turn gives us the following expression for the inverse operator Q^{-1}:

$$Q^{-1} = Q^2 + (q^2 + q^{-2} - 1)Q - (q^2 + q^{-2} - 1)\,\mathrm{Id}.$$

Now, the property (5.8) follows directly from relation (5.6) and explicit forms of $\mathcal{S}_q$ and Q^{-1} written above.

The second claim was actually proved in section 4, relation (4.20), where the construction (5.9) was found from the general approach to the mREA representation theory. But it can also be easily verified by direct calculations. To this end we rewrite the 'adjoint action' (5.9) in the basis of matrix copies $L_{\bar{k}}$. Then this action is given by relation (5.4)

$$L_{\bar{1}} \triangleright L_{\bar{2}} = L_1 R_{12} - R_{12} L_1 = (L_{\bar{1}} - L_{\bar{2}})R_{12},$$

where the symbol $\triangleright$ stands for the adjoint action.

This entails that

$$L_{\bar{2}} \triangleright L_{\bar{3}} = (L_{\bar{2}} - L_{\bar{3}})R_{23}. \tag{5.11}$$

Now, by writing the mREA commutation relation as

$$R_{12} L_{\bar{1}} L_{\bar{2}} - L_{\bar{1}} L_{\bar{2}} R_{12} = L_{\bar{2}} - L_{\bar{1}}$$

we apply the both sides of this equality to the element $L_{\bar{3}}$ and verify that the results are equal to each other. Indeed, we have

$$\begin{aligned} &R_{12} L_{\bar{1}} \triangleright (L_{\bar{2}} \triangleright L_{\bar{3}}) = (L_{\bar{2}} - L_{\bar{1}})R_{23} - R_{12}R_{23}R_{12}^{-1}(L_{\bar{2}} - L_{\bar{1}}), \\ &L_{\bar{1}} \triangleright (L_{\bar{2}} R_{12} \triangleright L_{\bar{3}}) = L_1 R_{12}^{-1} R_{23}^{-1} R_{12} - R_{12}^{-1} L_1 R_{23}^{-1} R_{12} - L_1 R_{23}^{-1} + R_{23}^{-1} R_{12}^{-1} L_1 R_{12} \end{aligned}$$

and

$$(L_{\bar{2}} - L_{\bar{1}}) \triangleright L_{\bar{3}} = L_{\bar{2}} R_{23} - R_{23} L_{\bar{2}} - R_{23} L_1 R_{12}^{-1} R_{23}^{-1} + R_{23} R_{12}^{-1} L_1 R_{23}^{-1}.$$

Then, taking the difference of the first two expressions and applying the Hecke condition to connect R and R^{-1}, we convince ourselves that the difference coincide with the third expression above, that is

$$R_{12}(L_{\bar{1}}\triangleright)(L_{\bar{2}}\triangleright) - (L_{\bar{1}}\triangleright)(L_{\bar{2}}\triangleright)R_{12} = (L_{\bar{2}}\triangleright) - (L_{\bar{1}}\triangleright).$$

The proof of the third claim of the theorem is a matter of a trivial calculation on the basis of (4.12), (5.4) and (5.11). Consider, for example, the action of the first relation (5.10) on the basis element $L_{\bar{1}} \dot{\otimes} L_{\bar{2}} \dot{\otimes} L_{\bar{3}}$ (we shall omit the symbols $\dot{\otimes}$ for simplicity). The right-hand side of the relation leads to the following chain of transformations:

$$L_{\bar{1}} L_{\bar{2}} L_{\bar{3}} \xrightarrow{(R_{\mathrm{End}})_{12}} L_{\bar{2}} L_{\bar{1}} L_{\bar{3}} \xrightarrow{(R_{\mathrm{End}})_{23}} L_{\bar{2}} L_{\bar{3}} L_{\bar{1}} \xrightarrow{[\,,\,]_{12}} (L_{\bar{2}} - L_{\bar{3}})R_{23} L_{\bar{1}},$$

while the left-hand side gives

$$L_{\bar{1}} L_{\bar{2}} L_{\bar{3}} \xrightarrow{[\,,\,]_{23}} L_{\bar{1}}(L_{\bar{2}} - L_{\bar{3}})R_{23} \xrightarrow{R_{\mathrm{End}}} (L_{\bar{2}} - L_{\bar{3}}) L_{\bar{1}} R_{23}$$

which coincides with the above result for the right-hand side since $L_{\bar{1}}$ commutes with the matrix R_{23}. □

We keep the notation $gl(V, R)$ for the following data:

$$(\mathcal{L} = \mathrm{span}(L_i^j), Q : \mathcal{L}^{\otimes 2} \to \mathcal{L}^{\otimes 2}, [\,,\,] = \circ(\mathrm{Id} - Q))$$

which is similar to that above related to an involutive R. We call this data the *braided Lie algebra* of *gl*-type.

Now, consider its sl-reduction. For this purpose, we pass to the mREA generators F_i^j and ℓ introduced in (3.20). Recall that this passage requires $\mathrm{Tr}(C) \neq 0$. The commutation relations of the new mREA generators are given by (3.21). It is not difficult to calculate the adjoint action in terms of F and ℓ:

$$\begin{aligned}
&\mathrm{ad}\,\ell(\ell) = 0, \qquad \mathrm{ad}\,\ell(F_1) = -(q - q^{-1})\mathrm{Tr}(C)F_1\\
&\mathrm{ad}\,F_{\bar{1}}(\ell) = 0,\\
&\mathrm{ad}\,F_{\bar{1}}(F_{\bar{2}}) = F_1R_{12} - R_{12}F_1 + (q - q^{-1})R_{12}F_1R_{12}^{-1}.
\end{aligned} \tag{5.12}$$

As was defined in section 3, the sl-reduction of the mREA $\mathcal{L}(R_q, 1)$ consists in passing to the quotient algebra $\mathcal{SL}(R_q, 1)$ (3.22). The explicit commutation relations of the $\mathcal{SL}(R_q, 1)$ algebra read

$$R_{12}F_1R_{12}F_1 - F_1R_{12}F_1R_{12} = (R_{12}F_1 - F_1R_{12}), \qquad \mathrm{Tr}_R(F) = 0. \tag{5.13}$$

Let us consider the space

$$\mathcal{SL} = \mathrm{span}\{F_i^j\} \subset \mathcal{L} = \mathrm{span}\{L_i^j\}$$

formed by the traceless elements with respect to the categorical trace tr_R (4.9):

$$\mathcal{SL} = \{X \in \mathcal{L} | \mathrm{tr}_R(X) = 0\}.$$

The restriction $Q \mapsto Q_{sl} : \mathcal{SL}^{\otimes 2} \to \mathcal{SL}^{\otimes 2}$ can also be naturally defined. It is easy to see that the categorical trace of the bracket (5.4) is zero, so the bracket can be restricted to the subspace $\mathcal{SL}$. However, the corresponding 'adjoint action' of the restricted bracket following from (5.13)

$$\mathrm{ad}_{sl}F_{\bar{1}}(F_{\bar{2}}) = [F_{\bar{1}}, F_{\bar{2}}]_{sl} = F_1R_{12} - R_{12}F_1$$

does not define a representation of $\mathcal{SL}$ because it does not contain the last term of formula (5.12).

Thus, if similarly to the braided Lie algebra $gl(V, R)$ we define the algebra $sl(V, R)$ as the following data:

$$(\mathcal{SL} = \mathrm{span}(F_i^j), Q_{sl} : \mathcal{SL}^{\otimes 2} \to \mathcal{SL}^{\otimes 2}, [\,,\,]_{sl} : \mathcal{SL}^{\otimes 2} \to \mathcal{SL}),$$

where $[\,,\,]_{sl}$ is the bracket (5.6) restricted to $\mathcal{SL}$, we can see that the Jacobi identity in the form valid for the former algebra fails for the algebra $sl(V, R)$.

A particular case of such a braided Lie algebra related to $U_q(sl(n))$ was introduced in [LS] (see the previous section). A similar construction was suggested in [DGHZ].

In our approach we need no object of QG type. Moreover, our construction is valid for Hecke symmetries of general type. In particular, it embraces q-deformations of Lie super-algebras $gl(m|n)$.

Quantum Lie algebras related to the QG $U_q(\mathfrak{g})$ of other series were introduced in [DGG]. Let us explain the main idea of that construction. Fix a Lie algebra $\mathfrak{g}$ and consider a $U_q(\mathfrak{g})$-covariant analog $[\,,\,]_q$ of the classical Lie bracket $[\,,\,] : \mathfrak{g}^{\otimes 2} \to \mathfrak{g}$. For this end we decompose the $U_q(\mathfrak{g})$-module $\mathfrak{g}^{\otimes 2}$ into a direct sum of irreducible $U_q(\mathfrak{g})$-modules. Here we assume the space $\mathfrak{g}$ to be endowed with a $U_q(\mathfrak{g})$ action deforming the usual adjoint action. By means of the QG coproduct this action can be extended onto $\mathfrak{g}^{\otimes 2}$. In contrast with the previous case the $U_q(\mathfrak{g})$-module $\mathfrak{g}^{\otimes 2}$ is multiplicity free. So, the bracket $[\,,\,]_q$ can be defined in a unique (up to a factor) way similarly to their classical counterparts but in the category of $U_q(\mathfrak{g})$-modules. Thus, a 'quantum (braided) Lie algebra' bracket can be introduced.

However, we know no reasonable Jacobi identity which could be written for such a 'quantum Lie algebra'. Moreover, the above 'enveloping algebra' does not possess the good deformation property. The point is that even the 'symmetric algebra' corresponding to such an 'enveloping algebra' is not a deformation of its classical counterpart.

As was mentioned in [GPS3], we think that an axiomatic introduction of a generalized (quantum, braided) Lie algebra is possible iff the corresponding braiding is involutive. Nevertheless, in some papers such algebras related to non-involutive braidings are introduced by the same (or very close) axiom system. However, this way of introducing the Lie algebra type objects is not justified by exhibiting meaningful examples.

To conclude this section, we would like to discuss the problem of defining a conjugation (involution) in a generalized (quantum, braided) Lie algebra. Let us first assume that $R : V^{\otimes 2} \to V^{\otimes 2}$ is a skew-invertible real involutive symmetry (so, $\mathbb{K} = \mathbb{R}$). Also, suppose that there exists a nondegenerated pairing $V^{\otimes 2} \to \mathbb{R}$ which is R-invariant.

Then the space $\mathrm{End}(V)$ can be identified with $V^{\otimes 2}$ and this identification is a categorical morphism in terminology of section 4. Let us introduce an involution in $\mathrm{End}(V)$ as the image of the operator $R : V^{\otimes 2} \to V^{\otimes 2}$. By passing to the complexification of the algebra $\mathrm{End}(V)$ we complete this operator with the complex conjugation of numerical coefficients. The final operator

$$* : \mathrm{End}(V) \to \mathrm{End}(V), \qquad x \to x^*$$

is called *conjugation*. It can be naturally extended to the enveloping algebra $U(gl(V, R))$ via the relation $(x \circ y)^* = \circ(* \otimes *)R(x \otimes y)$ where $\circ$ is the product in $U(gl(V, R))$.

This conjugation is involutive, R_{End}-invariant and it is coordinated with the bracket of generalized Lie algebra $gl(V, R)$ in the following sense:

$$[x, y]^* = -[x^*, y^*]. \tag{5.14}$$

Note that this relation is universal: it does not depend on a skew-invertible involutive symmetry R. The latter relation entails that the family of elements which are skew-symmetric with respect to the conjugation $x^* = -x$ is a subalgebra. Also, the map $x \to -x^*$ is an isomorphism of the generalized Lie algebra $gl(V, R)$.

Note that if R is the usual twist and the pairing is Euclidean the subalgebra of skew-symmetric elements is just $u(n)$.

Now, let V be a vector space endowed with a Hecke symmetry. We are interested in a problem of classification of all involutions (conjugations) in the algebra $\mathrm{End}(V)$ which are R-invariant and verifying (5.14). In the $U_q(sl(2))$ case it can be shown by a direct calculation that the only R-invariant linear operators on the space $\mathrm{End}(V)$ are scalar on each irreducible component in the decomposition $\mathrm{End}(V) = \mathbb{K} \oplus \mathcal{SL}$ where $\mathcal{SL}$ is the subspace of the traceless elements. So, the only involution on $\mathcal{SL}$ which is R-invariant operator reads $x \to x^* = -x$. It is out of interest and is not a deformation of the involution on the algebra $sl(2, \mathbb{C})$ giving rise to the algebra $su(2)$. We would like to emphasize that using in algebras in question involutions which are not categorical morphisms is not motivated by their 'braided nature' and leads to some shortcoming (see the following section).

6. The quantum sphere: different approaches

In this section, we consider different approaches to introducing the quantum sphere (hyperboloid) and developing some aspects of geometry on it. Historically, the RTT algebra was the first defined quantum matrix algebra. So, quantum homogeneous space algebras (and in the first turn, the quantum sphere) were initially defined by imitating the classical definition of homogeneous G-spaces as cosets G/H.

However, as explained in section 2 on a symmetric orbit there exists another way of defining the corresponding 'quantum variety' via quotienting the standard (m)REA. Let us compare these ways on the example of a quantum sphere. Though in this case the both ways lead to equivalent results, they yield completely different methods of adjacent geometry, in particular, those of constructing projective modules over q-spheres. Now, describe a quantum sphere explicitly.

First, describe the quantum algebra $\mathbb{K}_q[SL(2)]$ which is a quantum counterpart of the Sklyanin bracket on the group $SL(2)$. Computing the defining relations for the entries of the matrix $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we get

$$ab = qba, \quad ac = qca, \quad bd = qdb, \quad bc = cb, \quad cd = qdc,$$
$$ad - da = (q - q^{-1})bc, \quad ad - qbc = da - q^{-1}bc = 1.$$

Consider the subalgebra of this algebra consisting of elements invariant with respect to the coaction

$$\Delta : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a \otimes z & b \otimes z^{-1} \\ c \otimes z & d \otimes z^{-1} \end{pmatrix}$$

where z is a formal invertible indeterminate. We shall denote this subalgebra $\mathbb{K}_q[SL(2)/H]$. It is a q-deformation of the algebra of functions $\mathbb{K}[SL(2)/H]$.

The algebra $\mathbb{K}_q[SL(2)]$ can be equipped with the following conjugation operator:

$$a^* = d, \qquad b^* = -qc, \qquad c^* = -q^{-1}b, \qquad d^* = a, \tag{6.1}$$

which is assumed to be antilinear and subject to the usual condition

$$(xy)^* = y^*x^*, \qquad \forall x, y \in \mathbb{K}_q[SL(2)].$$

Consequently, it is involutive. Here we assume $\mathbb{K} = \mathbb{C}$ and q to be real.

Note that the Hecke symmetry coming in the definition of the algebra $\mathbb{K}_q[SL(2)]$ is a particular case of the so-called braiding of real type as defined in [M4] (definition 4.2.15). In our normalization of a Hecke symmetry R this condition reads $\bar{R}^{\mathsf{T}} = R$. If R is such a Hecke symmetry it is possible to introduce an involution in the corresponding RTT algebra in a similar way.

It is not difficult to see that the subalgebra $\mathbb{K}_q[SL(2)/H]$ is closed with respect to the conjugation (6.1). The algebra $\mathbb{K}_q[SL(2)/H]$ equipped with this involution will be denoted $\mathbb{K}_q[SU(2)/H]$. Namely, this algebra is usually considered as one of avatars of a quantum sphere.

Being equipped with the above conjugation (6.1), the algebra $\mathbb{K}_q[SL(2)]$ is treated to be a quantum counterpart of the algebra $\mathbb{K}[SU(2)]$. We denote it as $\mathbb{K}_q[SU(2)]$. All its $*$-representations (i.e. those respecting the $*$-operator) in a Hilbert space were classified in [VS], where it was shown that there is a series of one-dimensional representations and an infinite-dimensional one. They can be restricted to the subalgebra $\mathbb{K}_q[SL(2)/H]$.

Now, we consider a way of defining the quantum sphere (hyperboloid) via the REA. To this end consider the mREA $\mathcal{L}(R_q, \hbar)$ related to the standard Hecke symmetry. This algebra is explicitly given by the system (4.3) or (4.4) in the generators ℓ, h, b, c. Also, consider its quotient $\mathcal{SL}(R_q, \hbar)$ defined by the system (4.5). We shall refer to the algebra $\mathcal{SL}(R_q, \hbar)$ as *q-noncommutative* if $\hbar \neq 0$ and as *q-commutative* if $\hbar = 0$. Note, that since the algebras $\mathcal{L}(R_q, \hbar)$ and $\mathcal{L}(R_q)$ are isomorphic for $q \neq \pm 1$ we avoid to call them in a similar manner by keeping this terminology for their sl-quotients which are not isomorphic to each other for any value of q.

Consider the central element

$$\mathrm{Cas} = q^2\,\mathrm{Tr}_q L^2 = q^{-1}a^2 + q^{-1}bc + qcb + qd^2 = \frac{\ell^2}{2_q} + q^{-1}bc + \frac{h^2}{2_q} + qcb \in \mathcal{L}(R_q, \hbar).$$

Its image in the algebra $\mathcal{SL}(R_q, \hbar)$ reads

$$\mathrm{Cas}_{sl} = q^{-1}bc + \frac{h^2}{2_q} + qcb.$$

It is also central in this algebra. So, it is natural to introduce the following quotient:

$$\mathcal{SL}^C(R_q, \hbar) = \mathcal{SL}(R_q, \hbar)/\langle \mathrm{Cas}_{sl} - C\rangle,$$

where $C \in \mathbb{K}$ is a number. We assume that $C \neq 0$.

In what follows the elements Cas and Cas_{sl} are called the *q-Casimir elements* in the algebras $\mathcal{L}(R_q, \hbar)$ and $\mathcal{SL}(R_q, \hbar)$, respectively. Note that the center of the algebra $\mathcal{L}(R_q, \hbar)$ is generated by the elements Cas_{sl} and ℓ.

The algebra $\mathcal{SL}^C(R_q, \hbar)$ is called the *quantum hyperboloid* since at $q = 1$ we get just a usual hyperboloid (one or two sheeted in dependence on $C \in \mathbb{K} = \mathbb{R}$). By a *quantum sphere* one often means this algebra but considered over the field of complex numbers ($\mathbb{K} = \mathbb{C}$) and endowed with a conjugation (involution) defined on the generators as follows:

$$\ell^* = \ell, \qquad h^* = h, \qquad b^* = c, \qquad c^* = b.$$

Onto the whole algebra this involution is extended via the classical properties.

Nevertheless, such an involution does not allow us to define a quantum sphere as a real algebra. This is a shortcoming of the involution which is not a categorical morphism whereas a quantum hyperboloid can be treated as a real algebra if $q \in \mathbb{R}$.

Considering irreducible representations V_k, $\dim V_k = k + 1$ of the algebra $\mathcal{SL}(R_q, 1)$ we can see that the constant C coming in the definition of the algebra $\mathcal{SL}^C(R_q, 1)$ depends on k. The situation is similar to the case of the algebra $U(sl(2))$ in which the value of the Casimir element depends on k as well. By direct computation we get [S]

$$C(k) = C(V_k) = q^{-2}\frac{k_q(k+2)_q}{(k+1)_q}\frac{((k+2)_q + k_q)}{((k+2)_q - k_q)^2}.$$

As for the algebra $\mathcal{SL}^C(R_q) = \mathcal{SL}^C(R_q, 0)$, besides one-dimensional representations, it possesses two Verma-type representations which differ from one another by the sign. Finding these representations is left to the reader.

Now, we want to discuss different ways of quantizing vector bundles on the sphere. According to the Serre–Swan approach any vector bundle on an affine algebraic or a smooth variety can be realized as a finitely generated projective module over the coordinate algebra of the variety. As follows from [R], along with a formal deformation $A \to A_\hbar$ of a commutative algebra A any projective A-module M can also be deformed into an $A_\hbar$-module $M_\hbar$. Otherwise stated, the idempotent $e(M)$ corresponding to the module M can be formally deformed into an idempotent $e(M_\hbar)$ with entries belonging to the algebra $A_\hbar$.

Nevertheless, we deal with a non-formal deformation and our deformation parameters can be specialized. So, we want to get explicit expressions for idempotents over a quantum sphere (hyperboloid). Construction of such idempotents in the framework of the first approach was done in [HM]. In that paper a series of idempotents $e_{\pm 1}, e_{\pm 2}, \ldots$ was constructed which define the left $\mathbb{K}_q[SU(2)/H]$ modules $\mathbb{K}_q[SU(2)/H]^{\oplus(|k|+1)}e_k$. Let us reproduce two of these idempotents:

$$e_{-1} = \begin{pmatrix} ad & -q^{-1}ab \\ cd & -q^{-1}cb \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}(d, -q^{-1}b), \qquad e_1 = \begin{pmatrix} da & -qdc \\ ba & -qbc \end{pmatrix} = \begin{pmatrix} d \\ b \end{pmatrix}(a, -qc).$$

(These formulae differ from those of [HM] by replacing $q \to q^{-1}$.)

The authors of [HM] used this explicit realization of 'quantum line bundles' in order to compute the Chern–Connes index in a particular case: the projective module corresponding to $k = -1$ and a representation of the quantum sphere constructed in [MNW]. Recall that the index is defined by the pairing of (the class of) a representation π of a given algebra A and (the class of) a projective A-module M in accordance with the rule

$$\mathrm{Ind}(\pi, e) = \mathrm{Tr}(\pi(\mathrm{Tr}(e))), \tag{6.2}$$

where e is an idempotent corresponding to M.

Another approach to constructing projective modules over braided varieties (orbits) was suggested in [GS1, GLS2]. Also, in [GLS2] a q-analog of the Chern–Connes index was introduced and computed on q-spheres. The main tool employed in this approach is a series of the Cayley–Hamilton identities which are valid for the matrix $L_{(1)} = L$ and for its higher analogs $L_{(k)}$, $k \geqslant 2$. Here L is the matrix of the mREA generators and $L_{(k)}$ is the $(k+1)\times(k+1)$-dimensional matrix whose explicit form is given in [GLS2].

By passing to the algebra $\mathcal{SL}(R_q, \hbar)$ we get reductions of the matrices $L_{(k)} \to F_{(k)}$ and the Cayley–Hamilton identities for them. Being reduced to the algebra $\mathcal{SL}^C(R_q, \hbar)$ these identities take the form $p_k(F_{(k)}) = 0$ where p_k is a polynomial with numerical coefficients. Assuming the roots of the polynomial p_k to be distinct, we can associate with it $k+1$ idempotents $e_i(k)$, $i = 1, 2, \ldots, k+1$ with entries belonging to the algebra $\mathcal{SL}^C(R_q, \hbar)$.

Consider an example: the idempotents arising from the matrix L itself. In the algebra $\mathcal{SL}(R_q, \hbar)$ the matrix L reduces to

$$F = \begin{pmatrix} \frac{qh}{2_q} & b \\ c & -\frac{q^{-1}h}{2_q} \end{pmatrix} = b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + h\begin{pmatrix} \frac{q}{2_q} & 0 \\ 0 & -\frac{q^{-1}}{2_q} \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{6.3}$$

This matrix satisfies the Cayley–Hamilton identity

$$F^2 - q^{-1}\hbar F - \frac{\mathrm{Cas}_{sl}}{2_q}\,\mathrm{Id} = 0. \tag{6.4}$$

(This is an sl-reduction of the Cayley–Hamilton identity for the initial non-reduced form of the matrix L.) While we pass to the algebra $\mathcal{SL}^C(R_q, \hbar)$ in the space V_k the coefficient $\frac{\mathrm{Cas}_{sl}}{2_q}$ becomes $\frac{C}{2_q}$. Assuming the roots μ_i, $i = 0, 1$ of the equation

$$\mu^2 - q^{-1}\hbar\mu - \frac{C}{2_q} = 0$$

to be distinct we define two idempotents

$$e_i(1) = \frac{L - \mu_i\,\mathrm{Id}}{\mu_i - \mu_j}, \qquad i, j = 0, 1,\ j \neq i.$$

The q-index introduced in [GLS2] has a form similar to (6.2) but the trace Tr is replaced by its braided analog Tr_R

$$\mathrm{Ind}_q(\pi, e) = \mathrm{Tr}_R(\pi(\mathrm{Tr}_R e)).$$

Note that the elements $\mathrm{Tr}_R e_i(k)$ are central whereas it is not true for $\mathrm{Tr} e_i(k)$ if $q \neq 1$. Also, observe that this method of constructing projective modules over braided orbits is valid for the mREA related to skew-invertible Hecke symmetries of general type (but the case of non-even symmetries is less studied).

The following proposition was proved in [GLS2].

Proposition 18. *Let k be a sufficiently large positive integer. Then, with a proper numbering of the idempotents* $e_i(m)$, $0 \leqslant i \leqslant m$ *we have*

$$\mathrm{Ind}(\pi_k, e_i(m)) = (m + k - 2i + 1)_q,$$

where π_k *is the (right) R-invariant representation of the mREA* $\mathcal{L}(R_q, 1)$ *in the space* V_k. *Here we assume the R-trace to be normalized so that it is an additive–multiplicative functional on the corresponding Schur–Weyl category (see [GLS1]).*

We emphasize that we are dealing with a q-noncommutative quantum (braided) hyperboloid algebra. On setting $q = 1$, we get a noncommutative hyperboloid algebra which is a quotient of $U(sl(2))$. However, since the value of the Casimir element (both in the classical and quantum cases) depends on the space where the algebra in question is represented we are somewhat dealing with a series of (braided) hyperboloids. Namely, in the formula for the q-index the idempotent $e_i(m)$ has entries belonging to the algebra $\mathcal{L}^C(R_q, 1)$ with $C = C(V_k)$.

Note that the modules

$$M_m = e_0(m)\mathcal{L}^C(R_q, 1)^{\oplus(m+1)} \quad \text{and} \quad M_{-m} = e_m(m)\mathcal{L}^C(R_q, 1)^{\oplus(m+1)}, \quad m = 1, 2, \ldots$$

correspond to the line bundles $\mathcal{O}(m)$ and $\mathcal{O}(-m)$ respectively over the projective space $\mathbb{CP}^1$ (also, we put $M_0 = \mathcal{L}^C(R_q, 1)$, this module corresponds to the trivial line bundle). We refer the reader to [GLS2, GS2] for detail.

Now, we consider a quantum analog of the cotangent vector bundle on the hyperboloid also presented as a projective module. First, consider the usual hyperboloid

$$H^2 = \left\{b, h, c \in sl(2)^* | bc + \frac{h^2}{2} + cb = C \neq 0\right\}.$$

The space $\bigwedge^1(H^2)$ of 1-forms on it consists of all linear combinations

$$db\alpha + dh\beta + dc\gamma \qquad \alpha, \beta, \gamma \in \mathbb{K}[H^2]$$

modulo the submodule

$$\left(dbc + \frac{dhh}{2} + dcb\right)\varphi, \varphi \in \mathbb{K}[H^2].$$

Hereafter $\mathbb{K}[H^2] = \mathbb{K}[\mathbb{R}^3]/\langle 2bc + \frac{h^2}{2} - C\rangle$ is the coordinate ring of the hyperboloid H^2. So, we realize the space $\bigwedge^1(H^2)$ as a right $\mathbb{K}[H^2]$-module but since the algebra $\mathbb{K}[H^2]$ is commutative it can be endowed with a two-sided module structure. It is not difficult to show that this module is projective.

By passing to a q-analog $\mathbb{K}_q[H^2] = \mathcal{SL}(R_q)/\langle q^{-1}bc + \frac{h^2}{2_q} + qcb - C\rangle$ of the algebra $\mathbb{K}[H^2]$ we naturally define a q-analog of the space $\bigwedge^1(H^2)$ as a quotient module

$$\bigwedge\nolimits_q^1(H^2) = \mathbb{K}_q[H^2]^{\oplus 3}/M',$$

where $M' = e'\mathbb{K}_q[H^2]^{\oplus 3}$ is the right $\mathbb{K}_q[H^2]$-module such that the corresponding idempotent is

$$e' = \frac{1}{C}\left(q^{-1}c, \frac{h}{2_q}, qb\right)^{\mathsf{T}}(b, h, c).$$

Otherwise stated, in the right $\mathbb{K}_q[H^2]$-module $\bigwedge_q^1(H^2)$ the relation

$$q^{-1}dbc + \frac{dhh}{2_q} + qdcb = 0 \tag{6.5}$$

is imposed. It is also possible to realize the space $\bigwedge_q^1(H^2)$ as a left module $\mathbb{K}_q[H^2]^{\oplus 3}/M''$ where $M'' = \mathbb{K}_q[H^2]^{\oplus 3}e''$ and

$$e'' = \frac{1}{C}(c, h, b)^{\mathsf{T}}\left(q^{-1}b, \frac{h}{2_q}, qc\right). \tag{6.6}$$

This means that in the space of braided differentials we impose the relation

$$q^{-1}bdc + \frac{hdh}{2_q} + qcdb = 0.$$

Similarly to (6.5) this relation follows from the q-Casimir: in order to get these relations we replace the left (resp., right) factors in each summand of the q-Casimir by their differentials.

Emphasize that we are only dealing with one-sided modules without endowing them with a two-sided module structure. In order to convert such a one-sided module into a two-sided one we have to introduce a transposition of 'functions' and 'differentials' arising from the initial braiding R. However, as was shown in [AG2] there is no such a transposition which would ensure the good deformation property of the module.

In a similar way, we can treat the space $\bigwedge_q^2(H^2)$ of quantum 2-differentials as a one-sided $\mathbb{K}_q[H^2]$ module. Note that the same approach is valid on a q-noncommutative hyperboloid (sphere) or its classical analog $q = 1$ (see [AG1]). Observe that the Leibnitz rule for the differential d is not applicable in all these cases. However, it is possible to construct a q-analog of the classical de Rham operator by analyzing the decomposition of the spaces $\bigwedge_q^i(H^2)$, $i = 1, 2$ into a direct sum of irreducible $U_q(sl(2))$-module (and similarly for other algebras) and defining this q-analog via the classical pattern (see [AG1] for detail).

7. Differential calculus via Koszul complexes

In this section, we discuss the role of Koszul type complexes in constructing the differential calculus on the quantum matrix algebras.

Let V be a vector space over the ground field $\mathbb{K}$, $T(V) = \bigoplus_{k=0}^{\infty} V^{\otimes k}$ be its free tensor algebra and $I \subset V^{\otimes 2}$ be a vector subspace. Consider the quadratic algebra $A = T(V)/\langle I\rangle$ and introduce the following subspaces:

$$I^{\cap n} = V^{\otimes(n-2)} \otimes I \bigcap V^{\otimes(n-3)} \otimes I \otimes V \bigcap V^{\otimes(n-4)} \otimes I \otimes V^{\otimes 2} \bigcap \dots \bigcap I \otimes V^{\otimes(n-2)} \subset V^{\otimes n}, \quad n \geqslant 3.$$

Also we set by definition $I^{\cap 2} = I$, $I^{\cap 1} = V$ and $I^{\cap 0} = \mathbb{K}$.

Then the Koszul complex is defined by the chain of maps

$$\dots \to I^{\cap n} \otimes A \to I^{\cap(n-1)} \otimes A \to \dots \to I \otimes A \to V \otimes A \to A \to \mathbb{K} \to 0$$

where $A \to \mathbb{K}$ is the counit and for $n \geqslant 1$ the differential $d : I^{\cap n} \otimes A \to I^{\cap(n-1)} \otimes A$ reads

$$d\big(y_{i_1}y_{i_2}\dots y_{i_n} \otimes x\big) = y_{i_1}y_{i_2}\dots y_{i_{n-1}} \otimes y_{i_n}x \quad \forall x \in A, y_{i_1}, \dots, y_{i_n} \in V.$$

In fact, this complex splits up into disjoint complexes

$$\dots \to I^{\cap n} \otimes A^{(m)} \to I^{\cap(n-1)} \otimes A^{(m+1)} \to \dots \to I \otimes A^{(m+n-2)} \to V \otimes A^{(m+n-1)} \to A^{(m+n)} \to 0$$

where $A^{(m)} = V^{\otimes m}/I^{\cup m}$ and

$$I^{\cup m} = V^{\otimes (m-2)} \otimes I + V^{\otimes (m-3)} \otimes I \otimes V + V^{\otimes (m-4)} \otimes I \otimes V^{\otimes 2} + \cdots + I \otimes V^{\otimes (m-2)} \subset V^{\otimes m}, \quad m \geqslant 2.$$

Also, we put $A^{(1)} = V$ and $A^{(0)} = \mathbb{K}$. The algebra A is called *Koszul* if these complexes are acyclic for $m + n \geqslant 1$.

Assume now that in the space $V^{\otimes 2}$ there exists a subspace I_+ complimentary to I (which in the following will be denoted by I_-) such that for a given $m \geqslant 3$ the subspaces $I_-^{\cup m} \subset V^{\otimes m}$ and $I_+^{\cap m} \subset V^{\otimes m}$ are complimentary, i.e.

$$I_-^{\cup m} \bigcap I_+^{\cap m} = \{0\} \qquad \text{and} \qquad I_-^{\cup m} \oplus I_+^{\cap m} = V^{\otimes m}. \tag{7.1}$$

Let $P_+^{(m)} : V^{\otimes m} \to I_+^{\cap m}$ be the projector taking $I_-^{\cup m}$ to 0. Then for any element x of the homogeneous component $A^{(m)}, m \geqslant 2$, we have $x = P_+^{(m)}(x) \in I_+^{\cap m}$ modulo $I_-^{\cup m}$. If I_+ (respectively I_-) is subspace of symmetric (respectively skew-symmetric) elements then the projector $P_+^{(m)}$ is the operator of the complete symmetrization of elements from $A^{(m)}$. Presenting any element of the algebra A as a sum of homogeneous components we can symmetrize it as well. In this case we say that the element is presented in the *canonical form*.

Now, consider a family of quadratic algebras $A(\nu) = T(V)/\langle I(\nu)\rangle$ where $I(\nu) \subset V^{\otimes 2}$ is a subspace depending on a parameter ν. Since we are dealing with a non-formal deformation, we assume that $I(\nu)$ is defined by a system of finite linear combinations of generators of $V^{\otimes 2}$ with coefficients analytically depending on ν in a neighborhood U of 0. So, in U the parameter ν can be specialized. We are interested in such families of algebras that $\dim A^{(m)}(\nu) = \dim A^{(m)}$ for any integer m (at least for a generic $\nu \in U$), where $A = A(0)$.

Assume that for a family $I_-(\nu) = I(\nu)$ there exists another one $I_+(\nu)$ such that for any $m \geqslant 2$ the subspaces $I_-^{\cup m}(\nu)$ and $I_+^{\cap m}(\nu)$ are complementary and the projectors $P_+^{(m)}(\nu)$ analytically depend on $\nu \in U$ except may be a finite set of points which does not include 0. Then for a generic $\nu \in U$ we have $\dim A^{(m)}(\nu) = \dim A^{(m)}$ for all m, i.e. the algebra A have the good deformation property.

Given families $I_\pm(\nu)$ and therefore the projector $P_+^{(2)}(\nu)$, we look for a higher projector $P_+^{(m)}(\nu), m \geqslant 3$ as a polynomial in $\left(P_+^{(2)}(\nu)\right)_{12}, \ldots, \left(P_+^{(2)}(\nu)\right)_{m-1m}$ (the subscripts indicate the spaces in the product $V^{\otimes m}$ where the operator acts) with analytical coefficients. Having constructed such a polynomial, we can conclude that for a generic ν $\dim A^{(m)}(\nu) = \dim A^{(m)}$ (here 'a generic ν' means 'all ν except for a countable set'). If such projectors exist for all $m \geqslant 3$ we conclude that this property is valid for all homogeneous components of the algebra $A(\nu)$.

If in addition the initial algebra A is Koszul, then this is also true for the algebra $A(\nu)$ for a generic ν (see [PP]).

In examples below we have two families of subspaces $I_\pm(\nu) \subset V^{\otimes 2}$ and we want to show that the both algebras

$$A_+(\nu) = T(V)/\langle I_-(\nu)\rangle, \qquad A_-(\nu) = T(V)/\langle I_+(\nu)\rangle$$

have the good deformation property. In order to show this, we should construct the projectors $P_\pm^{(m)}(\nu), m \geqslant 3$ in terms of the operators $P_\pm^{(2)}(\nu)$. The 'skew-symmetrization' operators $P_-^{(m)}(\nu)$ are defined similarly to the projectors $P_+^{(m)}(\nu)$, provided that the subspaces $I_+^{\cup m}(\nu)$ and $I_-^{\cap m}(\nu)$ are complementary to each other for any $m \geqslant 2$. We call a couple of the subspaces $I_\pm(\nu)$ *regular* if the subspaces $I_\pm^{\cup m}(\nu)$ and $I_\mp^{\cap m}(\nu)$ are complementary for any $m \geqslant 2$.

In [G2] this scheme was applied to the algebras $\mathrm{Sym}_q(V)$ and $\bigwedge_q(V)$ related to the Hecke symmetries. Namely, a series of projectors $P_\pm^{(m)}(q)$ related to $I_-(q) = \mathrm{Im}(q\,\mathrm{Id} - R_q)$ and

$I_+(q) = \mathrm{Im}(q^{-1}\,\mathrm{Id} + R_q)$ was constructed. As follows from [G2], these algebras have the good deformation property. Note that if a family of Hecke symmetries is not quasiclassical the monomials $x_1^{a_1}\ldots x_n^{a_n}$ do not form a basis in the algebra $\mathrm{Sym}_q(V)$. So, the method of verifying the good deformation property of this algebra based on the ordering the generators is not valid any more, whereas the above scheme is still applicable.

Again, consider the algebras $\mathrm{Sym}_q(\mathcal{T})$ and $\mathcal{L}(R_q) = \mathrm{Sym}_q(\mathcal{L})$ defined by formulae (1.6) and (1.7) respectively (we do not assume a Hecke symmetry R to be quasiclassical). This means that the corresponding subspaces $I_-(\mathcal{T})$ and $I_-(\mathcal{L})$ are determined by the left-hand side of these formulae. Now, define the complementary 'symmetric' components $I_+(\mathcal{T})$ and $I_+(\mathcal{L})$ by putting

$$I_+(\mathcal{T}) = RT_1T_2 + T_1T_2R^{-1}, \qquad I_+(\mathcal{L}) = RL_1RL_1 + L_1RL_1R^{-1}$$

and define the corresponding algebras

$$\textstyle\bigwedge_q(\mathcal{T}) = T(\mathcal{T})/\langle I_+(\mathcal{T})\rangle, \qquad \bigwedge_q(\mathcal{L}) = T(\mathcal{L})/\langle I_+(\mathcal{L})\rangle. \tag{7.2}$$

Note that the projector $P_+^{(2)}(\mathcal{L})$ is nothing but the operator $\mathcal{S}_q$ discussed in section 5. In the paper [GPS3], an attempt was undertaken to construct the higher projectors $P_\pm^{(m)}$ via those $P_\pm^{(2)}$ in order to apply the above scheme. We have only succeeded in constructing these projectors for $m = 3$. (In fact, the construction is valid for all quantum matrix algebras associated with a compatible pair of Hecke symmetries, in particular, for $\mathrm{Sym}_q(\mathcal{T})$.) Nevertheless, as follows from [Dr2] this property suffices for concluding that for any $m \geqslant 4$ and a generic $q = e^\nu$ the dimensions of all homogeneous components are stable (classical for a quasiclassical R). So, according to [Dr2] we should only control the dimensions of the third homogeneous components. However, it would be interesting to explicitly construct the higher projectors.

Consider one example more, where this scheme can be hopefully applied, namely, the algebra $\mathcal{SL}(R_q) = \mathcal{L}(R_q)/\langle\ell\rangle$ (an analogous quotient of the algebra $\mathrm{Sym}_q(\mathcal{T})$ cannot be defined since there is no central element in $\mathcal{T}$). As we observed above the algebra $\mathcal{SL}(R_q)$ is defined by the same formulae as that $\mathcal{L}(R_q)$ but with the generators F_i^j instead of L_i^j. This means that the subspace $I_-(\mathcal{SL})$ corresponding to this algebra can be obtained by replacing the generators L_i^j in $I_-(\mathcal{L})$ by their traceless components F_i^j defined by formula (3.20).

It is more difficult to define the result of 'sl-reduction' of the algebra $\bigwedge_q(\mathcal{L})$. Describe this procedure by mainly following [IP1].

In accordance with definition (7.2) the algebra $\bigwedge_q(\mathcal{L})$ is generated by the elements L_i^j subject to the following multiplication rules:

$$R_{12}L_1R_{12}L_1 + L_1R_{12}L_1R_{12}^{-1} = 0. \tag{7.3}$$

We make a linear change (3.20) $L_i^j \to \{F_i^j, \ell\}$ including explicitly the R-trace $\ell = \mathrm{Tr}_R(L)$ into the set of generators. Of course, as well as in the case of mREA, we assume $\mathrm{Tr}_R(\mathrm{Id}) = \mathrm{Tr}(C) \neq 0$.

Calculating the R-trace in the second space of the defining relations (7.3) we find

$$\ell L + L\ell + \omega L^2 = 0, \qquad \omega = q - q^{-1}$$

or

$$\ell F + F\ell + \frac{\omega}{1 + \omega(\mathrm{Tr}_R(\mathrm{Id}))^{-1}}F^2 = 0 \tag{7.4}$$

that is the R-trace ℓ is not central in our algebra. As was shown in [IP1], the element ℓ is nilpotent: $\ell^2 = 0$.

Now we can find the multiplication table of the algebra $\bigwedge_q(\mathcal{L})$ in terms of the generators F and ℓ. Substituting the expression of L through F and ℓ into (7.3) and using the anticommutation relation (7.4) and nilpotency of ℓ, we get the following result:

$$\begin{aligned} &R_{12}F_1R_{12}F_1 + F_1R_{12}F_1R_{12}^{-1} = \kappa\big(F_1^2 + R_{12}F_1R_{12}\big),\\ &\ell F + F\ell = -\tau F^2,\\ &\ell^2 = 0, \qquad \mathrm{Tr}_R(F) = 0, \end{aligned} \tag{7.5}$$

where the numeric parameters κ and τ read

$$\kappa = \frac{\omega}{\mathrm{Tr}_R(\mathrm{Id}) + \omega}, \qquad \tau = \kappa\,\mathrm{Tr}_R(\mathrm{Id}).$$

Note that the R-traceless elements F_i^j form a *subalgebra* of $\bigwedge_q(\mathcal{L})$ since the element ℓ does not enter their multiplication table contrary to the case of mREA (see relations (3.21)). So, in this case we have no need to pass to a quotient algebra in order to obtain the sl-reduction—the traceless algebra is a subalgebra of the initial one. Finally, we put

$$I_+(\mathcal{SL}) = \mathrm{span}\big(R_{12}F_1R_{12}F_1 + F_1R_{12}F_1R_{12}^{-1} - \kappa\big(F_1^2 + R_{12}F_1R_{12}\big)\big)$$

and $\bigwedge_q(\mathcal{SL}) = T(\mathcal{SL})/\langle I_+(\mathcal{SL})\rangle$.

Hopefully, the couple of subspaces $I_\pm(\mathcal{SL})$ is regular. We are able to prove this claim if the initial Hecke symmetry is of the Temperley–Lieb type. A proof will be given in our subsequent paper.

Here we consider an example arising from the standard Hecke symmetry. The space $I_-(q = e^\nu)$ is determined by the left-hand side of relations (4.5). It is the spin 1 $U_q(sl(2))$-submodule of the space $\mathcal{SL}^{\otimes 2}$ endowed with an $U_q(sl(2))$-action. The corresponding algebra is just $\mathcal{SL}(R_q)$. The space $I_+(q) = V_2 \oplus V_0$ in this case is a direct sum of the spin 0 and spin 2 $U_q(sl(2))$-submodules of $\mathcal{SL}^{\otimes 2}$. Here

$$\begin{aligned} V_0 &= \mathrm{span}\left(q^{-1}b\otimes c + \frac{1}{2_q}h\otimes h + qc\otimes b\right),\\ V_2 &= \mathrm{span}(b\otimes b, q^2b\otimes h + h\otimes b, q^3b\otimes c - qh\otimes h + q^{-1}c\otimes b, q^2h\otimes c + c\otimes h, c\otimes c). \end{aligned} \tag{7.6}$$

It is not difficult to see that the braiding in the space $\mathcal{SL}^{\otimes 2}$ which is the extension of the initial Hecke symmetry is a BMW symmetry. With the use of methods of [OP2] it is possible to construct the projectors $P_\pm^{(m)}$. This is the basic idea of the aforementioned proof.

Remark 19. Given a subset $I_- \subset V^{\otimes 2}$ it is not clear whether there is a subspace I_+ such that the couple (I_-, I_+) is regular. But even if it is the case, the complementary subspace I_+ is not in general unique. Consider an example.

Let $R : V^{\otimes 2} \to V^{\otimes 2}$, $(\dim V = 2)$ be an involutive symmetry given by

$$\begin{aligned} &R(x\otimes x) = x\otimes x, \quad R(x\otimes y) = bx\otimes x + y\otimes x, \quad R(y\otimes x) = -bx\otimes x + x\otimes y,\\ &R(y\otimes y) = abx\otimes x - ax\otimes y + ay\otimes x + y\otimes y, \end{aligned}$$

where $\{x, y\}$ is a basis in V and $a, b \in \mathbb{K}$. Then $I_- = \mathrm{span}(-bx^2 + xy - yx)$ and $I_+ = \mathrm{span}(x^2, xy + yx, -axy + y^2)$ (we omit the sign $\otimes$). The couple (I_-, I_+) is regular for any $a, b \in \mathbb{K}$. Even if we put $b = 0$, i.e. if we consider the 'classical' skew-symmetric subspace I_-, the space I_+ is not unique and depends on a.

Assume (I_-, I_+) to be a regular couple. Consider the algebras $A_+ = A = T(V)/\langle I_-\rangle$ and $A_- = T(V)/\langle I_+\rangle$ and associate with them two Koszul complexes

$$d_- : A_-^{(m)} \otimes A_+^{(n)} \to A_-^{(m+1)} \otimes A_+^{(n-1)}, \qquad d_+ : A_-^{(m)} \otimes A_+^{(n)} \to A_-^{(m-1)} \otimes A_+^{(n+1)},$$

where we identify $A_+^{(n)}$ and $I_+^{\cap n}$ (resp., $A_-^{(m)}$ and $I_-^{\cap m}$). In [G2] these complexes associated with a Hecke symmetry were called Koszul complexes of the first kind. Now, we want to use them in order to define a differential and 'partial derivatives' on quantum matrix algebras.

Introduce a de Rham–Koszul differential d_{RK} on the component $A_-^{(m)} \otimes A_+^{(n)}$ by setting $d_{\mathrm{RK}} = nd_-$ (the factor n is motivated by an analogy with the classical case). Thus, we have

$$d_{\mathrm{RK}}\big(y \otimes x_{i_1}x_{i_2}\ldots x_{i_n}\big) = nyx_{i_1} \otimes x_{i_2}\ldots x_{i_n}, \qquad y \in A_-^{(m)}, x_{i_1}, x_{i_2}, \ldots, x_{i_n} \in V.$$

In particular, if $m = 0$ we treat this operator to be an analog of the de Rham differential on the algebra $A_+ = \bigoplus A_+^{(n)}$.

Now, define *braided partial derivatives* on this algebra in the standard way. If $f \in A_+^{(n)}$ is a homogeneous element, consider its image $d_{\mathrm{RK}}(f) = \sum_i x_i \otimes f_i$ where $f_i \in A_+^{(n-1)}$. Then we introduce the braided partial derivative in x_i by putting $\partial_{x_i}(f) = f_i$. The differential d_{RK} and the braided partial derivatives can be extended on the whole algebra A_+ by linearity. Below, we use the notation $\partial_i^j = \partial_{L_j^i}$ for the braided derivatives in the generators of the REA. Thus, we have $\partial_i^j L_k^l = \delta_k^j \delta_i^l$. Below we shall omit the term 'braided'.

This way of introducing the partial derivatives on quantum algebra does not use any form of the Leibnitz rule. Emphasize that we are only dealing with one-sided A_+-modules without transposing 'functions' and 'differentials'. Instead, we apply the de Rham–Koszul differential and the partial derivatives to elements presented in the canonical form. So, we do not need to verify any compatibility of the differential with the defining relations of the algebras in question. Below we use the same principle in order to define other 'braided vector fields'.

Again, let (I_-, I_+) be a regular couple of subspaces of $V^{\otimes 2}$. Now, consider another complex (called the Koszul complex of the second kind in [G2]). To this end we need the space V^* dual to V. This means that there exists a nondegenerated pairing $V^* \otimes V \to \mathbb{K}$. We extend this pairing on the spaces $(V^*)^{\otimes k}$ and $V^{\otimes k}$ by the rule

$$\langle a \otimes b, c \otimes d\rangle = \langle b, c\rangle\langle a, d\rangle, \qquad a, b \in V^*, c, d \in V$$

and so on. Define the subspaces $I_-^* = (I_+)^\perp$ and $I_+^* = (I_-)^\perp$ where $I^\perp \subset (V^*)^{\otimes 2}$ stands for the space orthogonal to $I \subset V^{\otimes 2}$. Also, we put

$$A_+^* = T(V^*)/\langle I_-^*\rangle \qquad \text{and} \qquad A_-^* = T(V^*)/\langle I_+^*\rangle.$$

Observe that the couple (I_-^*, I_+^*) is regular and present all elements of the algebras A_+ and A_-^* in the canonical form. Now introduce a differential

$$\tilde{d} : A_-^{*\,(m)} \otimes A_+^{(n)} \to A_-^{*\,(m-1)} \otimes A_+^{(n-1)}, \quad m, n \geqslant 1$$

via the pairing $\langle\, , \rangle_{m,m+1}$. This means that the last factor of $(A_-^*)^{(m)}$ and the first factor of $A_+^{(n)}$ are coupled. It is not difficult to see that $\tilde{d}^2 = 0$.

In what follows we consider the operator $\delta = n\tilde{d}$ (we have renormalized the initial differential by the same reason as above). In [G2], it was shown that the Koszul complex of the second kind associated with a skew-invertible Hecke symmetry is acyclic for a generic q. Hopefully, it is also so for the complexes associated with all quantum matrix algebras in question.

In the following section, we use the both kinds of Koszul type complexes in order to define q-analogs of wave operators.

Now, compare our definition of the derivatives on the algebra $\mathcal{L}(R_q)$ with that from [Me1, Me2]. To this end, besides the operator Q (5.5) we also introduce Q'

$$Q'(L_{\bar 1} \dot{\otimes} L_{\bar 2}) = R_{12}^{-1} L_{\bar 1} \dot{\otimes} L_{\bar 2} R_{12}^{-1}.$$

Then we have

$$I_-(\mathcal{L}) = \mathrm{Im}(\mathrm{Id} - Q), \qquad I_+(\mathcal{L}) = \mathrm{Im}(\mathrm{Id} + Q').$$

It is clear that these operators commute with each other. Also, the both operators satisfy the quantum Yang–Baxter equation, i.e. they are braidings. Besides, the couples $\{Q, Q'\}$ and $\{Q', Q\}$ are compatible in the sense of the definition of section 3 and $(\mathrm{Id} - Q)(\mathrm{Id} + Q') = 0$. For such a couple of operators it is possible to apply the scheme from [M1] where some aspects of the differential calculus on a space V endowed with a skew-invertible Hecke symmetry were developed[16]. Namely, let W be a vector space such that in $W^{\otimes 2}$ there are defined two operators Q and Q' satisfying the above conditions. Then the partial derivatives in the algebra $\mathrm{Sym}_q(W) = T(W)/\langle \mathrm{Im}(\mathrm{Id} - Q)\rangle$ can be defined as follows:

$$\partial^i x_j = \delta^i_j, \qquad \partial^i (x_j x_k) = \partial^i_1 (\mathrm{Id} + Q')(x_j x_k),$$
$$\partial^i (x_j x_k x_l) = \partial^i_1 (\mathrm{Id} + Q'_{12} + Q'_{12} Q'_{23})(x_j x_k x_l)$$

and so on. Here, $\{x_i\}$ is a basis of W and ∂^i_1 stands for the derivative in the generator x_i applied to the first factor.

Observe that we do not assume the braidings Q and Q' to be skew invertible. We do not need this property since we do not transpose the derivatives and the generators x_i. So, this way of proceeding differs from the usual Leibnitz rule.

Now, by assuming $W = \mathcal{L}$ apply these partial derivatives to a degree n homogeneous element $f \in \mathcal{L}^{(n)}(R_q)$ written in the canonical form. Then we have

$$(\mathrm{Id} + Q'_{12} + Q'_{12} Q'_{23} + \cdots + Q'_{12} Q'_{23} \cdots Q'_{n-1n})(f) = nf.$$

This follows from the fact that

$$Q'_{k-1k} P^{(n)}_+ = P^{(n)}_+ Q'_{k-1k} = P^{(n)}_+, \qquad 2 \leqslant k \leqslant n,$$

where $P^{(n)}_+$ is the 'symmetrization' projector. Note that this relation follows from the minimal polynomial for the operator Q' (which can be found similarly to that for Q) and the fact that a bigger projector 'absorb' a smaller one:

$$\big(P^{(2)}_+\big)_{k-1k} P^{(n)}_+ = P^{(n)}_+ \big(P^{(2)}_+\big)_{k-1k} = P^{(n)}_+, \qquad 2 \leqslant k \leqslant n.$$

This implies that our definition of partial derivatives and that from [Me1] are equivalent.

Let us go again to a general space W endowed with operators Q and Q' and define a q-analog of the de Rham differential in the same manner. Namely, we put

$$d(x_i x_j) = (dx_i) x_j + (d \otimes \mathrm{Id})(Q'(x_i x_j))$$

and so on. So, the space of one differentials is realized as a right module over the algebra $\mathrm{Sym}_q(W)$. Introducing the skew-symmetric algebra

$$\bigwedge\nolimits_q (W) = T(W)/\langle \mathrm{Im}(\mathrm{Id} + Q')\rangle$$

we treat the space $\bigwedge_q(W) \otimes \mathrm{Sym}_q(W)$ as the space of all differentials on W. An analog of the de Rham operator on this algebra can be introduced in the same way as above.

We complete this section with the following comment. The operators Q, Q', R_{End} appeared in the early 1990s (see [MMe] and references therein). They play a very important role in the q-analysis. In particular, each of the operators Q and Q' enables us to introduce the subspaces $I_\pm(\mathcal{L}) \subset \mathcal{L}^{\otimes 2}$ and therefore those $I_\pm(\mathcal{SL}) \subset \mathcal{SL}^{\otimes 2}$. However, it is not clear whether the latter subspaces can be defined via the operator R_{End}.

[16] Some aspects of such type calculus in a particular case related to a QG were previously considered in [WZ].

8. q-wave operators on q-Minkowski space algebra

As we noticed in section 1 the q-Minkowski space algebra was treated in some papers as a particular case of the REA. However, we want to slightly modify its definition in order to make it more similar to its classical counterpart.

There exists a family of quadratic, $U_q(sl(2))$-covariant algebras with central time variable $t = \ell$ which are deformations of $\mathbb{K}[\mathbb{R}^4]$. We get such algebras by replacing the factor $q - q^{-1}$ of the central element ℓ in the system (4.4) (with $\hbar = 0$) by an arbitrary multiplier α. A similar change can be done in the algebra $\mathcal{L}(R_q)$ related to any skew-invertible Hecke symmetry.

Let us now set $\alpha = 0$ and denote the corresponding algebra $\tilde{\mathcal{L}}(R_q)$. We call this algebra the *truncated REA*. In a similar way we define the *truncated mREA* $\tilde{\mathcal{L}}(R_q, \hbar)$. Explicitly, these algebras are defined as follows:

$$\tilde{\mathcal{L}}(R_q, \hbar) = \mathcal{SL}(R_q, \hbar) \otimes \mathbb{K}[\ell] \qquad \text{and} \qquad \tilde{\mathcal{L}}(R_q) = \mathcal{SL}(R_q) \otimes \mathbb{K}[\ell].$$

In the following, we denote the algebra $\tilde{\mathcal{L}}(R_q)$ in the $U_q(sl(2))$ case as $\mathbb{K}_q[\mathbb{R}^4]$ and consider it to be the q-Minkowski space algebra. Also, we put $\mathbb{K}_q[\mathbb{R}^3] = \mathbb{K}_q[\mathbb{R}^4]/\langle \ell \rangle$.

Thus, we have $\tilde{\mathcal{L}}(R_q) = T(\mathcal{L})/\langle \tilde{I}_- \rangle$ where

$$\tilde{I}_- = I_- \oplus \mathrm{span}(\ell \otimes b - b \otimes \ell, \ell \otimes h - h \otimes \ell, \ell \otimes c - c \otimes \ell) \subset \mathcal{L}^{\otimes 2}$$

and $I_- = I_-(q)$ is the subspace spanned by the left-hand side of (4.5). More explicitly, $\mathbb{K}_q[\mathbb{R}^4]$ is generated by four generators $\{\ell, h, b, c\}$ subject to the relations

$$q^2 hb - bh = 0, \qquad q^2 ch - hc = 0, \qquad 2_q q(bc - cb) + (q^2 - 1)h^2 = 0,$$
$$\ell f = f \ell, \qquad \forall f \in \mathrm{span}(b, h, c).$$

Besides, consider the subspace

$$\tilde{I}_+ = I_+ \oplus \mathrm{span}(\ell \otimes b + b \otimes \ell, \ell \otimes h + h \otimes \ell, \ell \otimes c + c \otimes \ell, \ell \otimes \ell) \subset \mathcal{L}^{\otimes 2},$$

where $I_+ = I_+(q) = V_0 \oplus V_2$ (see (7.6)).

Note that regularity of the couple (I_-, I_+) entails the same property for $(\tilde{I}_-, \tilde{I}_+)$. Therefore, we can apply the scheme of the previous section to the algebras $\mathbb{K}_q[\mathbb{R}^4]$ and $\bigwedge_q(\mathcal{L}) = T(\mathcal{L})/\langle \tilde{I}_+ \rangle$ and define the partial derivatives ∂_i^j on the algebra $\mathbb{K}_q[\mathbb{R}^4]$.

Also, we need the R-invariant pairing on the space $\mathcal{L}$. Such a pairing can be defined by the formula

$$\langle\, , \rangle : \mathcal{L}^{\otimes 2} \to \mathbb{K}, \qquad \left\langle L_i^j, L_k^l \right\rangle = \delta_i^l B_k^j, \tag{8.1}$$

where the operator B was introduced in (3.4). So, the space $\mathcal{L}$ can be identified with its dual.

However, such a pairing is not unique. It becomes unique (up to a factor) being restricted to the space $\mathcal{SL}$. In the standard case the pairing $\mathcal{SL} \otimes \mathcal{SL} \to \mathbb{K}$ can be chosen with the following normalization:

$$\langle b, c \rangle = q^{-1}, \qquad \langle h, h \rangle = 2_q, \qquad \langle c, b \rangle = q \tag{8.2}$$

(all other terms are trivial). Let us extend this pairing to the space $\mathcal{L}$ by putting

$$\langle \ell, \ell \rangle = \epsilon^{-1}, \qquad \langle \ell, f \rangle = \langle f, \ell \rangle = 0 \quad \forall f \in \mathcal{SL}, \qquad \epsilon \in \mathbb{K}, \epsilon \neq 0. \tag{8.3}$$

This is the most general form of a nondegenerated $U_q(sl(2))$-invariant pairing on the space $\mathcal{L}$.

Then one can extend this pairing to the spaces $\mathcal{SL}^{\otimes k} \otimes \mathcal{SL}^{\otimes k}$, $k \geqslant 2$ as was explained in the previous section. The following proposition can be proved by straightforward computations.

Proposition 20. *The spaces I_- and I_+ are orthogonal to each other with respect to the pairing (8.2). It is also true for the spaces $\tilde{I}_-$ and $\tilde{I}_+$ and the pairing (8.2)–(8.3).*

For the extended pairing we keep the same notation.

Definition 21. *We say that a triple* $(I_-, I_+ \subset V^{\otimes 2}, \langle\,,\,\rangle : V^{\otimes 2} \to \mathbb{K})$ *is regular if the couple* (I_-, I_+) *is regular and the subspaces* I_- *and* I_+ *are orthogonal to each other with respect to the pairing* $\langle\,,\,\rangle$.

So, proposition 20 states that the above triples are regular. This property enables us to identify the algebra A_- and that A_-^* in notations of the previous section and consequently to consider the differentials d_{RK} and δ as acting on the same terms $A_-{}^{(m)} \otimes A_+^{(n)}$. Thus, the operator

$$\delta d_{\mathrm{RK}} : A_-{}^{(m)} \otimes A_+^{(n)} \to A_-{}^{(m)} \otimes A_+^{(n-2)}$$

is well defined.

Also, this property of the subspaces $\tilde{I}_-$ and $\tilde{I}_+$ enables us to compute the commutation relations between the partial derivatives $\partial_\ell, \partial_h, \partial_b, \partial_c$ acting on the algebra $\mathbb{K}_q[\mathbb{R}^4]$. To this end we consider another basis $\{D_\ell, D_c, D_h, D_b\}$ in the space spanned by these derivatives. Namely, we put

$$D_\ell = \partial_\ell, \qquad D_c = q\partial_b, \qquad D_h = 2_q\partial_h, \qquad D_b = q^{-1}\partial_c.$$

Note that this basis is more convenient to deal with since the map $b \to D_b, h \to D_h, c \to D_c, \ell \to D_\ell$ is R-invariant whereas that $b \to \partial_b, \ldots$ is not. Explicitly, these operators are defined via the pairing:

$$D_b(f) = n\langle b, f\rangle_{12}, \qquad \text{if} \quad f \in \mathcal{L}^{(n)}(R_q) \quad \text{and so on,}$$

where the subscript means that the pairing with b is applied to the first factor of an element f.

The next proposition is a straightforward corollary of proposition 20.

Proposition 22. *The derivative* D_ℓ *commutes with* D_c, D_h, D_b. *The derivatives* D_c, D_h *and* D_b *satisfy the relations*

$$\begin{aligned} &q^2 D_h D_b - D_b D_h = 0 \\ &2_q q(D_b D_c - D_c D_b) + (q^2 - 1) D_h D_h = 0 \\ &q^2 D_c D_h - D_h D_c = 0. \end{aligned} \tag{8.4}$$

Otherwise stated, the map $\tau : b \to D_b, h \to D_h, c \to D_c, \ell \to D_\ell$ *is a representation of the algebra* $\mathbb{K}_q[\mathbb{R}^4]$.

This proposition entails that the operators

$$\Delta_{\mathbb{K}_q[\mathbb{R}^3]} = q^{-1} D_b D_c + \frac{D_h^2}{2_q} + q D_c D_b = q^{-1}\partial_c\partial_b + 2_q\partial_h^2 + q\partial_b\partial_c, \tag{8.5}$$

$$\Delta_{\mathbb{K}_q[\mathbb{R}^4]} = \epsilon D_\ell^2 + q^{-1} D_b D_c + \frac{D_h^2}{2_q} + q D_c D_b = \epsilon\partial_\ell^2 + q^{-1}\partial_c\partial_b + 2_q\partial_h^2 + q\partial_b\partial_c \tag{8.6}$$

are central. Here, ϵ is an arbitrary non-trivial factor. We call them q-Laplace operators on the algebras $\mathbb{K}_q[\mathbb{R}^3]$ and $\mathbb{K}_q[\mathbb{R}^4]$, respectively. Thus, our q-Laplace operator on the algebra $\mathbb{K}_q[\mathbb{R}^4]$ depends on ϵ.

Note that these operators are obtained from the quadratic central elements in the algebra $\mathbb{K}_q[\mathbb{R}^3]$ and $\mathbb{K}_q[\mathbb{R}^4]$ respectively in which we replaced the generators by the corresponding derivatives: $b \to D_b$ and so on.

Now, we pass to constructing q-Dirac operators on the algebras $\mathbb{K}_q[\mathbb{R}^3]$, $\mathbb{K}_q[\mathbb{R}^4]$ and the q-hyperboloid algebra $\mathbb{K}_q[H^2]$. All of these operators can be defined via a universal scheme making use of the *split* Casimir elements. Note that this scheme is similar to that from [GLS2] where a way of relating the REA Cayley–Hamilton identities and the split Casimir elements was suggested.

We begin with classical algebras. First, consider the $sl(2)$ *split* Casimir element

$$b\otimes c+\frac{h\otimes h}{2}+c\otimes b\in sl(2)\otimes sl(2).$$

Let π be the spin $\frac{1}{2}$ representation of the algebra $sl(2)$ in the standard basis. Applying this representation to the right factors of the split Casimir and the map τ (with $q=1$) defined in proposition 22 to the left factors we convert the split Casimir element into the 2×2 matrix. Its transposed matrix reads

$$D_b\begin{pmatrix}0&1\\0&0\end{pmatrix}+\frac{1}{2}D_h\begin{pmatrix}1&0\\0&-1\end{pmatrix}+D_c\begin{pmatrix}0&0\\1&0\end{pmatrix}=\begin{pmatrix}\frac{D_h}{2}&D_b\\D_c&-\frac{D_h}{2}\end{pmatrix}. \tag{8.7}$$

It can be treated as the Dirac operator on the space $\mathbb{R}^3\cong sl(2)^*$. It is easy to see that the square of this matrix is

$$\left(\frac{D_h^2}{4}+D_bD_c\right)\mathrm{Id}=\left(\partial_h^2+\partial_b\partial_c\right)\mathrm{Id}.$$

In order to get a similar operator on the space $\mathbb{R}^3\cong so(3)^*$ equipped with the Euclidean coordinates

$$x=\frac{\mathbf{i}(b+c)}{2},\qquad y=\frac{c-b}{2},\qquad z=\frac{\mathbf{i}(a-d)}{2}$$

we proceed in a similar way with the $so(3)$ split Casimir element $-2\mathbf{i}(x\otimes x+y\otimes y+z\otimes z)$ (all numerical factors are introduced for our convenience). Then we arrive to the operator

$$\partial_x\begin{pmatrix}0&1\\1&0\end{pmatrix}+\partial_y\begin{pmatrix}0&-\mathbf{i}\\\mathbf{i}&0\end{pmatrix}+\partial_z\begin{pmatrix}1&0\\0&-1\end{pmatrix}=\begin{pmatrix}\partial_z&\partial_x-\mathbf{i}\partial_y\\\partial_x+\mathbf{i}\partial_y&-\partial_z\end{pmatrix}. \tag{8.8}$$

Being squared it equals $\left(\partial_x^2+\partial_y^2+\partial_z^2\right)\mathrm{Id}$.

A passage to the four-dimensional Minkowski space can be done in the usual way (see formula (8.16)).

To obtain the Dirac operator on the sphere S^2 of the radius $r=1$ we replace the partial derivatives in (8.8) by the infinitesimal rotations

$$X=z\partial_y-y\partial_z,\qquad Y=x\partial_z-z\partial_x,\qquad Z=y\partial_x-x\partial_y.$$

Thus, we get the operator

$$\mathrm{Dir}_{\mathbb{K}[S^2]}=\begin{pmatrix}Z&X-\mathbf{i}Y\\X+\mathbf{i}Y&-Z\end{pmatrix}.$$

It satisfies the equation

$$\mathrm{Dir}^2_{\mathbb{K}[S^2]}=\mathbf{i}\,\mathrm{Dir}_{\mathbb{K}[S^2]}+(X^2+Y^2+Z^2)\,\mathrm{Id}.$$

In a similar way, on replacing the derivatives D_b, D_h, D_c in (8.7) by the hyperbolic infinitesimal rotations

$$B=-2b\partial_h+h\partial_c,\qquad H=2b\partial_b-2c\partial_c,\qquad C=-h\partial_b+2c\partial_h$$

we can introduce the Dirac operator on a hyperboloid (see formula (8.15)).

Now, we are going to use the same scheme for the braided split Casimir

$$q^{-1}b\otimes c+\frac{h\otimes h}{2_q}+qc\otimes b\in\mathcal{SL}\otimes\mathcal{SL}.$$

Applying the representation (4.7) to the right factors in this split Casimir we get a matrix whose transposed form coincides with the matrix (6.3) (up to a numerical multiplier). So, by definition, the q-Dirac operator on the algebra $\mathbb{K}_q[\mathbb{R}^3]$ is the matrix (6.3) where we assume that $\hbar=0$ and replace the left factors b,h,c by D_b,D_h,D_c respectively (i.e. we apply the representation τ). Finally, we have

$$\mathrm{Dir}_{\mathbb{K}_q[\mathbb{R}^3]}=D_b\sigma_b+D_h\sigma_h+D_c\sigma_c=\begin{pmatrix}\frac{qD_h}{2_q} & D_b\\ D_c & -\frac{D_h}{2_q q}\end{pmatrix}. \tag{8.9}$$

Here the matrices $\sigma_b,\sigma_h,\sigma_c$ are respectively the multipliers of b,h,c in formula (6.3).

Using the Cayley–Hamilton identity (6.4) for the matrix (6.3) (where we put $\hbar=0$) we find that

$$\mathrm{Dir}^2_{\mathbb{K}_q[\mathbb{R}^3]}=\frac{1}{2_q}\left(q^{-1}D_bD_c+\frac{1}{2_q}D_h^2+qD_cD_b\right)\mathrm{Id}.$$

Now, define the q-Dirac operator on the algebra $\mathbb{K}_q[\mathbb{R}^4]$ in the usual way by setting

$$\mathrm{Dir}_{\mathbb{K}_q[\mathbb{R}^4]}=\epsilon D_\ell\begin{pmatrix}0_2 & \mathrm{I}_2\\ \mathrm{I}_2 & 0_2\end{pmatrix}+D_b\begin{pmatrix}\sigma_b & 0_2\\ 0_2 & -\sigma_b\end{pmatrix}+D_h\begin{pmatrix}\sigma_h & 0_2\\ 0_2 & -\sigma_h\end{pmatrix}+D_c\begin{pmatrix}\sigma_c & 0_2\\ 0_2 & -\sigma_c\end{pmatrix}, \tag{8.10}$$

where 0_2 and I_2 are respectively the trivial and unit 2×2 matrices.

It is easy to see that

$$\mathrm{Dir}^2_{\mathbb{K}_q[\mathbb{R}^4]}=\left(\epsilon^2D_\ell^2+\frac{1}{2_q}\left(q^{-1}D_bD_c+\frac{1}{2_q}D_h^2+qD_cD_b\right)\right)\mathrm{Id}. \tag{8.11}$$

In order to introduce a q-Dirac operator on the q-hyperboloid we need braided analogs of the hyperbolic infinitesimal rotations. They are defined via a braided analog of the Lie algebra $sl(2)$. This analog can be introduced in frameworks of the general scheme discussed in section 5. However, since the $sl(2)$-module $sl(2)^{\otimes 2}$ is multiplicity free, this braided analog can be defined in a more simple way. Let $\mathcal{SL}$ be the space $sl(2)$ endowed with the action of the QG $U_q(sl(2))$ deforming the classical adjoint one. There exists a unique (up to a nontrivial factor) $U_q(sl(2))$-morphism

$$[\,,\,]:\mathcal{SL}\otimes\mathcal{SL}\to\mathcal{SL}.$$

Explicitly, it is given by the following multiplication table

$$\begin{gathered}[b,b]=0,\quad [b,h]=-wb,\quad [b,c]=w\tfrac{q}{2_q}h,\quad [h,b]=wq^2b,\\ [h,h]=w(q^2-1)h,\quad [h,c]=-wc,\quad [c,b]=-w\tfrac{q}{2_q}h,\quad [c,h]=wq^2c,\quad [c,c]=0,\end{gathered} \tag{8.12}$$

where $w\in\mathbb{K}$, $w\neq 0$ is an arbitrary factor.

The corresponding adjoint action

$$\mathrm{ad}x(y)=[x,y]\quad\forall x,y\in\mathcal{SL}$$

gives rise to three operators

$$B_q=\mathrm{ad}\,b,\qquad H_q=\mathrm{ad}\,h,\qquad C_q=\mathrm{ad}\,c. \tag{8.13}$$

In the basis $\{b, h, c\}$ they are represented by the following matrices:

$$B_q = w\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & \frac{q}{2_q} \\ 0 & 0 & 0 \end{pmatrix} \quad H_q = w\begin{pmatrix} q^2 & 0 & 0 \\ 0 & q^2-1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad C_q = w\begin{pmatrix} 0 & 0 & 0 \\ -\frac{q}{2_q} & 0 & 0 \\ 0 & q^2 & 0 \end{pmatrix}. \tag{8.14}$$

Note that these operators satisfy relations (4.5) with $\hbar = \frac{w(q^4-q^2+1)}{2_q}$. By specializing $q = 1$ and $w = 2$ we get the adjoint representation of the Lie algebra $sl(2)$.

Now, we want to extend the action of operators B_q, H_q and C_q to the higher components of the algebra $\mathbb{K}_q[\mathbb{R}^3]$. Such an extension can be constructed via the coproduct described in section 4. However, we use another method which is similar to that used in the definition of the partial derivatives. Presenting a degree n homogeneous element $f \in \mathbb{K}_q[\mathbb{R}^3]$ in the canonical form we define the action of the extended operators as follows:

$$B_q(f) = n(B_q)_1(f), \qquad H_q(f) = n(H_q)_1(f), \qquad C_q(f) = n(C_q)_1(f),$$

where as usual the subscript means that these operators are applied to the first factors. (We keep the same notation for the extended operators.)

Note that on each homogeneous component of the algebra $\mathbb{K}_q[\mathbb{R}^3]$ the operators B_q, H_q, C_q realize a representation of the algebra (4.5) but (in contrast with the classical case) with different factors $\hbar$. Let $\hbar(n)$ be the value of the factor $\hbar$ on the homogeneous component of the degree n. A proof of this fact and the computation of the values $\hbar(n)$ can be found in [DGR].

Now, upon replacing the operators D_b, D_h, D_c in (8.9) by the operators B_q, H_q, C_q, respectively we get the q-Dirac operator on the q-hyperboloid (more precisely, a free $\mathbb{K}_q[H^2]$-module). Namely, we have

$$\mathrm{Dir}_{\mathbb{K}_q[H^2]} = B_q\sigma_b + H_q\sigma_h + C_q\sigma_c = \begin{pmatrix} \frac{qH_q}{2_q} & B_q \\ C_q & -\frac{H_q}{2_q q} \end{pmatrix}. \tag{8.15}$$

In virtue of (6.4) this operator on the degree n homogeneous component satisfies the relation

$$\mathrm{Dir}^2_{\mathbb{K}_q[H^2]} = q\hbar(n)\,\mathrm{Dir}_{\mathbb{K}_q[H^2]} + \frac{1}{2_q}\left(q^{-1}B_qC_q + \frac{H_q^2}{2_q} + C_qB_q\right)\mathrm{Id}.$$

Now, we discuss a way of definition of a q-analog of the Maxwell operators on the quantum algebras in question. First, consider the classical Maxwell operator

$$\mathrm{Mw}(\omega) = \partial\mathrm{d}\omega = \Delta(\omega) - \mathrm{d}\partial(\omega), \qquad \text{where} \quad \omega \in \Omega^1,\ \partial = *^{-1}\mathrm{d}*,$$

d is the de Rham operator, and $*$ is the Hodge one. The Hodge operator is introduced via a metric which is assumed to be (quasi)Euclidian. Identifying the differential forms

$$\omega = dt\alpha + dx\beta + dy\gamma + dz\delta \in \Omega(\mathbb{R}^4), \qquad \alpha, \beta, \gamma, \delta \in \mathbb{K}[\mathbb{R}^4]$$

with the columns $(\alpha, \beta, \gamma, \delta)^{\mathsf{T}}$ we can present the Maxwell operator on the Minkowski space as follows:

$$\mathrm{Mw}_{\mathbb{K}[\mathbb{R}^4]}\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} \Delta_{\mathbb{K}[\mathbb{R}^4]}(\alpha) \\ \Delta_{\mathbb{K}[\mathbb{R}^4]}(\beta) \\ \Delta_{\mathbb{K}[\mathbb{R}^4]}(\gamma) \\ \Delta_{\mathbb{K}[\mathbb{R}^4]}(\delta) \end{pmatrix} - \begin{pmatrix} \partial_t \\ \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}(\partial_{\mathrm{t}} - \partial_{\mathrm{x}}, -\partial_{\mathrm{y}}, -\partial_z)\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}.$$

In a similar manner we can realize the Maxwell operator on the Euclidian spaces $\mathbb{R}^3 \cong so(3)^*$ and $\mathbb{R}^3 \cong sl(2)^*$ (see [DG] for detail).

Now, introduce q-analogs of the Maxwell operators on the algebras $\mathbb{K}_q[\mathbb{R}^3]$ and $\mathbb{K}_q[\mathbb{R}^4]$ respectively by the relations

$$\mathrm{Mw}_{\mathbb{K}_q[\mathbb{R}^3]}\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} = \begin{pmatrix}\Delta_{\mathbb{K}_q[\mathbb{R}^3]}(\alpha)\\ \Delta_{\mathbb{K}_q[\mathbb{R}^3]}(\beta)\\ \Delta_{\mathbb{K}_q[\mathbb{R}^3]}(\gamma)\end{pmatrix} - \begin{pmatrix}\partial_b\\ \partial_h\\ \partial_c\end{pmatrix}(q^{-1}\partial_c, 2_q\partial_h, q\partial_b)\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}$$

$$\mathrm{Mw}_{\mathbb{K}_q[\mathbb{R}^4]}\begin{pmatrix}\alpha\\ \beta\\ \gamma\\ \delta\end{pmatrix} = \begin{pmatrix}\Delta_{\mathbb{K}_q[\mathbb{R}^4]}(\alpha)\\ \Delta_{\mathbb{K}_q[\mathbb{R}^4]}(\beta)\\ \Delta_{\mathbb{K}_q[\mathbb{R}^4]}(\gamma)\\ \Delta_{\mathbb{K}_q[\mathbb{R}^4]}(\delta)\end{pmatrix} - \begin{pmatrix}\partial_b\\ \partial_h\\ \partial_c\\ \partial_\ell\end{pmatrix}(q^{-1}\partial_c, 2_q\partial_h, q\partial_b, \epsilon\partial_\ell)\begin{pmatrix}\alpha\\ \beta\\ \gamma\\ \delta\end{pmatrix}, \tag{8.16}$$

where $\alpha, \beta, \gamma \in \mathbb{K}_q[\mathbb{R}^3]$ (resp., $\alpha, \beta, \gamma, \delta \in \mathbb{K}_q[\mathbb{R}^4]$) and the operator $\Delta_{\mathbb{K}_q[\mathbb{R}^3]}$ (resp., $\Delta_{\mathbb{K}_q[\mathbb{R}^4]}$) is defined by formula (8.5) (resp., (8.6)).

The basic property of these operators is that their kernels are similar to those of the classical Maxwell operators. Namely, the kernel of the operator $\mathrm{Mw}_{\mathbb{K}_q[\mathbb{R}^3]}$ (resp., $\mathrm{Mw}_{\mathbb{K}_q[\mathbb{R}^4]}$) contains all columns of the form $(\partial_b\varphi, \partial_h\varphi, \partial_c\varphi)^{\mathsf{T}}$ (resp., $(\partial_b\varphi, \partial_h\varphi, \partial_c\varphi, \partial_\ell\varphi)^{\mathsf{T}}$). This property is a consequence of the fact that the q-Laplace operators are central in the corresponding algebras.

In order to get the q-Maxwell operator on the q-hyperboloid algebra we present the partial derivatives in a form similar to that expressing the classical operators in the spherical coordinates. We do not know any quantum analogs of the angle variables, but instead we use the tangent braided vector fields which are analogs of the infinitesimal hyperbolic rotations. First, consider the space $\mathbb{R}^3 \cong so(3)^*$. Let us present the partial derivatives in the Euclidean variables x, y, z as follows:

$$\partial_x = \frac{yZ - zY}{r^2} + \frac{x}{r}\partial_r = \frac{yZ - zY}{\rho} + 2x\partial_\rho, \qquad (c.p. \to \partial_y, \partial_z), \tag{8.17}$$

where $\rho = r^2 = (x^2 + y^2 + z^2)$ and $c.p.$ stands for the cyclic permutations $x \to y \to z \to x$.

Here, instead of the derivatives in the angles we use the vector fields X, Y, Z tangent to all spheres $S_r^2 = \{(x, y, z)|x^2 + y^2 + z^2 = r^2\}$. These tangent fields are bound by the relation

$$xX + yY + zZ = 0. \tag{8.18}$$

Besides, they commute with the derivatives ∂_r and ∂_ρ. Note that the derivative ∂_ρ acts on the Cartesian variables x, y, z as follows:

$$\partial_\rho x = \frac{x}{2\rho}, \qquad \partial_\rho y = \frac{y}{2\rho}, \qquad \partial_\rho z = \frac{z}{2\rho}. \tag{8.19}$$

In terms of the vector fields X, Y, Z and the derivative ∂_ρ the Laplace operator on $\mathbb{R}^3$ takes the following form:

$$\Delta_{\mathbb{K}[\mathbb{R}^3]} = \partial_x^2 + \partial_y^2 + \partial_z^2 = \frac{X^2 + Y^2 + Z^2}{\rho} + 6\partial_\rho + 4\rho\partial_\rho^2. \tag{8.20}$$

Taking into consideration this formula, it is reasonable to extend the algebra $\mathbb{K}[\mathbb{R}^4]$ by the element ρ^{-1} (and to proceed in a similar way with the other algebras considered below). For detail we refer the reader to [GS3].

In a similar way, we can proceed while dealing with $\mathbb{R}^3 \cong sl(2)^*$ endowed with a $sl(2)$-invariant metric. Then, using the hyperbolic infinitesimal rotations

$$B = -2b\partial_h + h\partial_c, \qquad H = 2b\partial_b - 2c\partial_c, \qquad C = -h\partial_b + 2c\partial_h$$

we can present the derivatives ∂_b, ∂_h and ∂_c in a form similar to (8.17) (see formula (8.22) for $q = 1$).

Note that the operators B, H, C are not independent but are bound by a relation analogous to (8.18)

$$bC + \frac{hH}{2} + cB = 0.$$

A similar statement is valid for the braided vector fields B_q, H_q and C_q acting on the algebra $\mathbb{K}_q[\mathbb{R}^3]$.

Proposition 23. *The operators B_q, H_q and C_q obey the relation*

$$q^{-1}bC_q + \frac{hH_q}{2_q} + qcB_q = 0. \tag{8.21}$$

Proof. It suffices to check this relation on the generators of the algebra $\mathbb{K}_q[\mathbb{R}^3]$. Taking into consideration our method of prolongation of the operators B_q, H_q and C_q to the higher components we can conclude that it remains true on the whole algebra $\mathbb{K}_q[\mathbb{R}^3]$. □

Note that relation (8.21) is independent on the normalization of the operators B_q, C_q and H_q on the higher homogeneous components of the algebra $\mathbb{K}_q[\mathbb{R}^3]$.

All combinations $\alpha B_q + \beta H_q + \gamma C_q$ where $\alpha, \beta, \gamma \in \mathbb{K}_q[\mathbb{R}^3]$ are called *the braided tangent vector fields*. The space of such fields is a left $\mathbb{K}_q[\mathbb{R}^3]$-module

$$M = \mathbb{K}_q[\mathbb{R}^3]^{\oplus 3}/\overline{M}, \qquad \text{where} \quad \overline{M} = \left\{\varphi\left(q^{-1}bC_q + \frac{hH_q}{2_q} + qcB_q\right)\middle|\forall\varphi \in \mathbb{K}_q[\mathbb{R}^3]\right\}.$$

On any q-hyperboloid algebra $\mathbb{K}_q[H^2]$ the modules $\overline{M}$ and M are projective. The module M is similar to $\bigwedge_q^1(H^2)$ though one of them is the right module and the other is the left one.

Also, we need the following q-analog of the variable ρ. We put

$$\rho_q = \frac{1}{2_q}\left(q^{-1}bc + \frac{h^2}{2_q} + qcb\right).$$

In analogy with the classical formulae, we introduce the derivative ∂_{ρ_q} setting by definition

$$\partial_{\rho_q} b = \frac{b}{2\rho_q}, \qquad \partial_{\rho_q} h = \frac{h}{2\rho_q}, \qquad \partial_{\rho_q} c = \frac{c}{2\rho_q}.$$

In addition, we assume that the derivative ∂_{ρ_q} is subject to the usual Leibnitz rule. It is easy to see that this way of introducing the derivative ∂_{ρ_q} is compatible with the defining relations of the algebra $\mathbb{K}_q[\mathbb{R}^3]$. Note that we impose the Leibnitz rule only on the derivative in a central element. However, here we can also do without this rule by applying the method above. Namely, assuming $f \in \mathbb{K}_q[\mathbb{R}^3]$ to be a homogeneous element of degree n we apply this derivative to its first factor and multiply the result by n.

Proposition 24. *The following operator equalities are valid on the algebra $\mathbb{K}_q[\mathbb{R}^3]$:*

$$D_b = \frac{q^{-2}}{2_q\rho_q}\mathcal{B}_q + \frac{2b}{2_q}\partial_{\rho_q}, \qquad D_h = \frac{q^{-2}}{2_q\rho_q}\mathcal{H}_q + \frac{2h}{2_q}\partial_{\rho_q}, \qquad D_c = \frac{q^{-2}}{2_q\rho_q}\mathcal{C}_q + \frac{2c}{2_q}\partial_{\rho_q}, \tag{8.22}$$

where $\mathcal{B}_q = q^2hB_q - bH_q$, $\mathcal{H}_q = q2_q(bC_q - cB_q) + (q^2-1)hH_q$, $\mathcal{C}_q = q^2cH_q - hC_q$ and in definition (8.13)—(8.12) of the operators B_q, H_q, C_q we put $w = 1$.

Analogously to the previous proposition it suffices to check these relations on the generators of the algebra $\mathbb{K}_q[\mathbb{R}^3]$. We call relations (8.22) the *pseudospherical form* of the derivatives D_b, D_h and D_c.

Now, define the q-Laplace operator on the q-hyperboloid with $\rho_q = q^{-2}2_q$ by

$$\Delta_{\mathbb{K}_q[H^2]} = q^{-1}\mathcal{B}_q\mathcal{C}_q + \frac{1}{2_q}\mathcal{H}_q^2 + q\mathcal{C}_q\mathcal{B}_q$$

and the q-Maxwell operator by

$$\mathrm{Mw}_{\mathbb{K}_q[H^2]}\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} = \mathrm{e}'\left(\begin{pmatrix}\Delta_{\mathbb{K}_q[H^2]}(\alpha)\\ \Delta_{\mathbb{K}_q[H^2]}(\beta)\\ \Delta_{\mathbb{K}_q[H^2]}(\gamma)\end{pmatrix} - \begin{pmatrix}q^{-1}\mathcal{C}_q\\ \frac{\mathcal{H}_q}{2_q}\\ q\mathcal{B}_q\end{pmatrix}(\mathcal{B}_\mathrm{q}, \mathcal{H}_\mathrm{q}, \mathcal{C}_\mathrm{q})\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}\right),$$

where $(\alpha, \beta, \gamma)^\mathsf{T} \in e'\mathbb{K}_q[H]^{\oplus 3}$, $e' = 1 - \bar{e}'$ and $\bar{e}'$ is defined by (6.6).

This method of defining the q-Laplace and the q-Maxwell operators on the q-hyperboloid algebra was suggested in [DG]. It is motivated by the classical case, where on a subvariety of an affine space these operators are restrictions of similar operators on the ambient space (see [DG] for detail).

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References

[A] D'Andrea F Noncommutatice geometry and quantum groups symmetries arXiv:0811.3187

[AG1] Akueson P and Gurevich D 1999 Some aspects of braided geometry: differential calculus, tangent space, gauge theory *J. Phys. A: Math. Gen.* **32** 4183–97

[AG2] Akueson P and Gurevich D 2000 Cotangent and tangent modules on quantum orbits *Int. J. Mod. Phys.* B **14** 2287–509

[AKR] de Azcárraga J, Kulish P and Ródenas F 1994 Reflection equation and q-Minkowski space algebras *Lett. Math. Phys.* **32** 173–82

[BG] Braverman A and Gaistgory D 1996 The Poincaré–Brkhoff–Witt theorem for quadratic algebras of Koszul type *J. Algebr.* **181** 315–28

[BK] Bibikov P and Kulish P 2000 Dirac operators on quantum $SU_q(2)$ and quantum sphere *J. Math. Sci.* **100** 2039–50 (Engl. Transl.)

[CSSW1] Carow-Watamura U, Schlieker M, Scholl M and Watamura S 1990 Tensor representation of the quantum group $SL_q(2, \mathbb{C})$ and the quantum Minkowski space *Z. Phys.* C **48** 159–65

[CSSW2] Carow-Watamura U, Schlieker M, Scholl M and Watamura S 1991 A quantum Lorentz group *Int. J. Mod. Phys.* A **6** 3081–108

[CP] Chari V and Pressley A 1994 *A Guide to Quantum Groups* (Cambridge: Cambridge University Press)

[C] Cherednik I 1984 Factorizing particles on a half line, and root systems *Theor. Math. Phys.* **61** 977–83 (Engl. Transl.)

[CHZ] Chu C-S, Ho P-M and Zumino B 1996 The quantum 2-sphere as a complex quantum manifold *Z. Phys.* C **70** 339–44

[DGH] Dabrowski L, Grosse H and Hajac P 2001 Strong connections and Chern–Connes pairing in the Hopf–Galois theory *Comput. Math. Phys.* **220** 301–31

[DGG] Delius G W, Gardner C and Gould M D 1998 The structure of quantum Lie algebra for the classical series B_l, C_l and D_l *J. Phys. A: Math. Gen.* **31** 1995–2019

[DGHZ] Delius G W, Gould M D, Hüffmann A and Zhang Y-Z 1996 Quantum Lie algebras associated to $U_q(gl(n))$ and $U_q(sl(n))$ *J. Phys. A: Math. Gen.* **29** 5611–7

[Dob] Dobrev V 1994 New q-Minkowski space-time and q-Maxwell equations hierarchy from q-conformal invariance *Phys. Lett.* B **341** 133–8

[DS] Dobrev V and Sudbery A 1998 Representations of the generalized Lie algebra $sl(2)_q$ *J. Phys. A: Math. Gen.* **31** 6635–45

[D] Donin J Double quantization on coadjoint representations of semisimple Lie groups and their orbits arXiv:QA/9909160

[DGK] Donin J, Gurevich D and Khoroshkin S 1998 Double quantization of $\mathbf{CP}^n$ type orbits by generalized Verma modules *J. Geom. Phys.* **28** 384–406

[DGR] Donin J, Gurevich D and Roubtsov V 1997 Quantum hyperboloid and braided modules *Algèbres Noncommutative, Groupes Quantiques et Invariants* (Societé Mathématique de France, Collection Seminar et Congrès N 2) 103–18

[DGS] Donin J, Gurevich D and Shnider S 1999 Double quantization on some orbits in the coadjoint representation of simple Lie groups *Comput. Phys. Math.* **204** 39–60

[DKM] Donin J, Kulish P P and Mudrov A I 2003 On universal solution to reflection equation *Lett. Math. Phys.* **63** 179–94

[DM] Donin J and Mudrov A 2002 Explicit equivariant quantization on (co)adjoint orbits of $GL(n, \mathbb{C})$ *Lett. Math. Phys.* **62** 17–32

[Dr1] Drinfeld V 1986 Quantum groups *Proc. ICM, Berkeley,* vol 1 AMS, pp 798–820

[Dr2] Drinfeld V 1992 On quadratic quasi-commutational relations in quasi-classical limit *Sel. Math. Sov.* **11** 317–26 (Engl. Transl.)

[DG] Dutriaux A and Gurevich D 2008 Maxwell operator on q-Minkowski space and q-hyperboloid algebra *J. Phys. A: Math. Theor.* **41** 315203

[FP] Faddeev L and Pyatov P 1996 The differential calculus on quantum linear groups *Trans. Am. Math. Soc.* **175** 35–47

[FM] Friedel L and Maillet J-M 1991 Quadratic algebras and integrable systems *Phys. Lett.* B **262** 278–84

[FRT] Reshetikhin N, Takhtadzhyan L and Faddeev L 1990 Quantization of Lie groups and Lie algebras *Leningrad Math. J.* **1** 193–225 (Engl. Transl.)

[GM] Gomez X and Majid S 2003 Braided Lie algebras and bicovariant differential calculi over co-quasitriangular Hopf algebras *J. Algebr.* **261** 334–88

[G1] Gurevich D 1983 Generalized translation operators on Lie groups *Sov. J. Contemp. Math. Anal.* **18** 57–90 (Engl. Transl.)

[G2] Gurevich D 1991 Algebraic aspects of the Yang–Baxter equation *Leningrad Math. J.* **2** 801–28

[G3] Gurevich D 1995 Braided modules and reflection equations *Quantum Groups and Quantum Spaces, Banach Center Publications* vol 40 (Warszawa: Institut of Mathematics, Polish Academy of Sciences) pp 99–110

[GLS1] Gurevich D, Leclercq R and Saponov P 2002 Traces in braided categories *J. Geom. Phys.* **44** 251–78

[GLS2] Gurevich D, Leclercq R and Saponov P 2005 q-Index on braided noncommutative spheres *J. Geom. Phys.* **53** 392–420

[GP] Gurevich D and Panyushev D 1994 On Poisson pairs associated with modified R-matrices *Duke Math. J.* **73** 249–55

[GPS1] Gurevich D, Pyatov P and Saponov P 2005 Cayley–Hamilton theorem for quantum matrix algebras of $GL(m|n)$ type *St. Petersburg Math. J.* **17** 157–79

[GPS2] Gurevich D, Pyatov P and Saponov P 2006 $GL(m|n)$ type quantum matrix algebras: II. The structure of the characteristic subalgebra and its spectral parametrization *Theor. Math. Phys.* **147** 14–46

[GPS3] Gurevich D, Pyatov P and Saponov P 2008 Representation theory of (modified) reflection equation algebra of the $GL(m|n)$ type *Algebr. Anal.* **20** 70–133 (Engl. Transl. in *St Petersburg Math. J.*)

[GS1] Gurevich D and Saponov P 2001 Quantum line bundles via Cayley–Hamilton identity *J. Phys. A: Math. Gen.* **34** 4553–69

[GS2] Gurevich D and Saponov P 2007 Geometry of non-commutative orbits related to Hecke symmetries *Contemp. Math.* **433** 209–50

[GS3] Gurevich D and Saponov P Wave operators on quantum algebras via noncanonical quantization *Acta Appl. Math.* to be published

[HM] Hajac P and Majid S 1999 Projective module description of the q-monopole *Comput. Math. Phys.* **206** 247–64

[H] Hlavaty L 1995 Algebraic framework for quantization of nonultralocal models *J. Math. Phys.* **36** 4882–97

[IOP] Isaev A, Ogievetsky O and Pyatov P 1999 On quantum matrix algebras satisfying the Cayley–Hamilton–Newton identities *J. Phys. A: Math. Gen.* **32** L115–L121

[IP1] Isaev A and Pyatov P 1995 Covariant differential complexes on quantum linear groups *J. Phys. A: Math. Gen.* **28** 2227–46

[IP2] Isaev A and Pyatov P 2009 Spectral extension of the quantum group cotangent bundle *Comput. Math. Phys.* **288** 1137–79

[Ka] Karolinsky 2000 A classification of Poisson homogeneous spaces of complex reductive Poisson–Lie groups *Banach Center Publ.* **51** 103–8 (Polish Acad. Sci. Warsaw)

[KRR] Khoroshkin S, Radul A and Roubtsov V 1993 A family of Poisson structures on Hermitian symmetric spaces *Comput. Math. Phys.* **152** 299–315

[K] Kulish P P 1995 Representations of q-Minkowski space algebra *St. Petersburg Math. J.* **6** 365–74

[KS] Kulish P and Sklyanin E 1992 Algebraic structure related to the reflection equation *J. Phys. A: Math. Gen* **25** 5963–75

[KSS] Kulish P P, Sasaki R and Schwiebert C 1993 Constant solutions of reflection equations and quantum groups *J. Math. Phys.* **34** 286–304

[L] Lu J-H 2000 Classical dynamical r-matrices and homogeneous Poisson–Lie structures on G/H and K/T *Comput. Math. Phys.* **212** 337–70

[LS] Lyubashenko V and Sudbery A 1998 Generalized Lie algebras of type A_n *J. Math. Phys.* **39** 3487–504

[M1] Majid S 1993 Free braided differential calculus, braided binomial theorem, and the braided exponential map *J. Math. Phys.* **34** 4843–56

[M2] Majid S 1994 Quantum and braided-Lie algebras *J. Geom. Phys.* **13** 307–56

[M3] Majid S 1996 Introduction to braided geometry and q-Minkowski space *Quantum Groups and Their Applications in Physics (Varenna, 1994)* (*Proc. Int. School Phys. Enrico Fermi* vol 127) (Amsterdam: IOS)

[M4] Majid S 1995 *Foundations of Quantum Group Theory* (Cambridge: Cambridge University Press)

[MM] Majid S and Markl M 1996 Glueing operation for R-matrices, quantum groups an link-invariants of Hecke type *Math. Proc. Camb. Soc.* **119** 139–66

[MMe] Majid S and Meyer U 1994 Braided matrix structure of q-Minkowski space and q-Poincaré group *Z. Phys.* C **63** 357–62

[Ma] Manin Yu 1988 *Quantum Groups and Non-Commutative Geometry* (CRM, Université de Montreal)

[MNW] Masuda T, Nakagami Y and Watanabe J Noncommutative differential geometry on the quantum two sphere of Podleś: 1. An algebraic viewpoint *K-Theory* **5** 151–75

[Me1] Meyer U 1995 q-Lorentz group and braided coaddition on q-Minkowski space *Comput. Math. Phys.* **168** 249–64

[Me2] Meyer U 1996 Wave equations on q-Minkowski space *Comput. Math. Phys.* **174** 457–75

[Mu1] Mudrov A 2002 Characters of $U_q(gl(m))$-reflection equation algebra *Lett. Math. Phys* **60** 283–91

[Mu2] Mudrov A 2007 On quantization of Semenovtian–Shansky Poisson bracket on simple algebraic groups *St. Petersburg Math. J.* **18** 797–808

[N] Nijenhuis A 1955 Jacobi-type identities for bilinear differential concomitants of certain tensor firelds: I, II *Nederl. Akad. Wetensch. Proc. A vol 58 Indag. Math.* **17** 390–403

[O] Ogievetsky O 2002 Uses of quantum spaces *Contemp. Math.* **294** 161–232

[OP1] Ogievetsky O and Pyatov P Lecture on Hecke algebras *Proc. International School 'Symmetries and Integrable Systems' Dubna, Russia, 8–11 June 1999. JINR, Dubna, D2,5-2000-218* pp 39–88 (*Preprint* CPT-2000/P.4076 and MPI 01-40)

[OP2] Ogievetsky O and Pyatov P *Orthogonal and Symplectic Quantum Matrix Algebras and Cayley–Hamilton Theorem for them* arXiv:QA/0511618

[OSWZ] Ogievetsky O, Schmidke W, Wess J and Zumino B q-deformed Poincaré algebra *Comput. Math. Phys.* **150** 495–518

[PS] Pinzul A and Stern A 2001 Dirac operator on the quantum sphere *Phys. Lett.* B **512** 217–24

[P1] Podleś P 1987 Quantum spheres *Lett. Math. Phys.* **14** 193–202

[P2] Podleś P 1989 Differential calculus on quantum spheres *Lett. Math. Phys.* **18** 107–19

[P3] Podleś P 1992 The classification of differential structures on quantum 2-spheres *Comput. Math. Phys.* **150** 167–79

[P4] Podleś P 1996 Solutions of Klein–Gordon and Dirac equations on quantum Minkowski spaces *Comput. Math. Phys.* **181** 569–85

[PP] Polishchuk A and Positselski L Quadratic Algebras *University Lecture Series* vol 37 (Providence, RI: American Mathematical Society)

[R] Rosenberg J Rigidity of K-theory under deformation quantization arXiv:QA/9607021

[S] Saponov P 2004 The Weyl approach to the representation theory of reflection equation algebra *J. Phys. A: Math. Gen.* **37** 5021–46

[Sch] Scheunert M 1979 Generalized Lie algebras *J. Math. Phys.* **20** 712–20

[SWZ] Schmidke W, Wess J and Zumino B 1991 A q-deformed Lorentz algebra *Z. Phys.* C **52** 471–6

[Sh] Sheu A 1991 Quantization of the Poisson $SU(2)$ and its Poisson homogeneous space—the 2-sphere *Comput. Math. Phys.* **135** 217–32 (with appendix by A Weinstein and J-H Lu)

[SSV] Shklyarov D, Sinel'shchikov S and Vaksman L 1999 A q-analogue of the Berezin quantization method *Lett. Math. Phys.* **49** 253–61

[T] Turaev V 1994 *Quantum Invariant of Knots and 3-Manifolds* (Berlin: de Gruyter)

[VS] Vaksman L and Soibelman Y 1988 An algebra of functions on the quantum group $SU(2)$ *Funct. Anal. Appl.* **22** 170–81

[WZ] Wess J and Zumino B 1990 Covariant differential calculus on the quantum hyperplane *Nucl. Phys.* B **18** 302–12

[W] Woronowicz S 1989 Differential calculus on compact matrix pseudogroups (quantum groups) *Comput. Math. Phys.* **122** 125–70